

From non-Kählerian surfaces to Cremona group of $\mathbb{P}^2(\mathbb{C})$

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Abstract

For any minimal compact complex surface S with $b_2(S) > 0$ containing global spherical shells (GSS) there exists a family of surfaces $\mathcal{S} \rightarrow B$ with GSS which contains as fibers S , some Inoue-Hirzebruch surface and non minimal surfaces, such that blown up points are generically effective parameters. These families are versal outside a non empty hypersurface $T \subset B$. In case of surfaces with a cycle and one tree of rational curves we give new normal forms of contracting germs in Cremona group $Bir(\mathbb{P}^2(\mathbb{C}))$ and show that they admit a birational structure. These families contain all possible surfaces, in particular all surfaces S with GSS and $0 < b_2(S) \leq 3$ admit a birational structure.

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1 Introduction

Hopf surfaces are defined by contracting invertible germs $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$. There are well-known normal forms

$$F(z_1, z_2) = (az_1 + tz_2^m, bz_2), \quad 0 < |a| \leq |b| < 1, \quad (a - b^m)t = 0, \quad m \in \mathbb{N}^*,$$

which give effective parameters of the versal deformation and give charts with transition mappings in the group $Aut(\mathbb{C}^2)$ of polynomial automorphisms of \mathbb{C}^2 , in particular in the Cremona group $Bir(\mathbb{P}^2(\mathbb{C}))$ of birational mappings of $\mathbb{P}^2(\mathbb{C})$. Hopf surfaces are particular cases of a larger family of compact complex surfaces in the VII_0 class of Kodaira, namely surfaces S containing global spherical shells (GSS). When $b_2(S) \geq 1$, these surfaces admit neither affine nor projective structures [17, 21, 18]. Their explicit construction consists in the composition Π of $n = b_2(S)$ blowing-ups (depending on $2n$ parameters) followed by a special glueing by a germ of isomorphism σ (depending on an infinite number of parameters). These surfaces are not almost homogeneous [26] hence $0 \leq \dim H^0(S, \Theta) \leq 1$ and Chern classes of surfaces in class VII_0 satisfy the conditions $b_2(S) = c_2(S) = -c_1^2(S)$. By Riemann-Roch formula, we obtain the dimension of the base of the versal deformation of S ,

$$2n \leq \dim H^1(S, \Theta) = 2b_2(S) + \dim H^0(S, \Theta) \leq 2n + 1,$$

where Θ is the sheaf of holomorphic vector fields. Some questions are raised

- (1) Are the $2n$ parameters of the blown up points effective parameters ?
- (2) If there are non trivial global vector fields, there is at least one missing parameter. How to choose it ?
- (3) Do compact surfaces with GSS admit a birational structure, i.e. is there an atlas with transition mappings in Cremona group $Bir(\mathbb{P}^2(\mathbb{C}))$. More precisely is there in each conjugation class of contracting germs of the form $\Pi\sigma$ (or of strict germs, following Favre terminology [13]) a birational representative ?

Known results:

- If S is a Enoki surface (see [8]) or a Inoue-Hirzebruch surface (see [6]) known normal forms, namely

$$F(z_1, z_2) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, tz_2), \quad 0 < |t| < 1,$$

and

$$N(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s),$$

respectively, are birational. Here $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Gl(2, \mathbb{Z})$ is the composition of matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with at least one of the second type.

- If S is of intermediate type (see definition in section 2), there are normal forms due to C.Favre [13]

$$F(z_1, z_2) = (\lambda z_1 z_2^s + P(z_2), z_2^k), \quad \lambda \in \mathbb{C}^*, s \in \mathbb{N}^*, k \geq 2,$$

where P is a special polynomial. These normal forms are adapted to logarithmic deformations and show the existence of a foliation, however *are not birational*. In [25] K.Oeljeklaus and M.Toma explain how to recover second Betti number which is now hidden and give coarse moduli spaces of surfaces with fixed intersection matrix,

- Some special cases of intermediate surfaces are obtained from Hénon mappings H or composition of Hénon mappings. More precisely, the germ of H at the fixed point at infinity is strict, hence yields a surface with a GSS [16, 10]. These germs are birational.

Motivation:

Let S be a minimal compact complex surface with Betti numbers $b_1(S) = 1$, $n = b_2(S) > 0$, the class of such surfaces will be denoted VII_0^+ . We consider the following conditions:

- (A) S contains a global spherical shell (GSS),
- (B) S contains $b_2(S)$ rational curves,
- (C) S contains a cycle of rational curves,
- (D) S admits a deformation into $b_2(S)$ times blown up Hopf surfaces.

Conjecture: *All these properties are equivalent, and any class VII_0^+ surface possesses a global spherical shell (GSS) i.e. an open submanifold biholomorphic to a standard neighborhood of S^3 in \mathbb{C}^2 which does not disconnect the surface.*

We have

$$(A) \iff (B) \implies (C) \implies (D)$$

In fact $(A) \implies (B)$ by the construction of GSS surfaces and $(B) \implies (A)$ by [11],

$(A) \implies (C)$ also by construction (see [5]) The implication $(C) \implies (D)$ has been obtained by I.Nakamura [23, 24].

The strategy developped in [27, 28] is aimed to show that any surface in VII_0^+ satisfies condition (C), therefore the solution to the following problem would be a step toward the conjecture:

Problem: Let $\mathcal{S} \rightarrow \Delta$ be a family of compact surfaces over the disc such that for every $u \in \Delta^*$, S_u contains a GSS. Does S_0 contain a GSS ? In other words, are the surface with GSS closed in families ?

To solve this problem we have to study families of surfaces in which curves do not fit into flat families, the volume of some curves in these families may be not uniformly bounded (see [12]) and configurations of curves change. Favre normal forms of polynomial germs associated to surfaces with GSS, cannot be used because the discriminant of the intersection form is fixed. Moreover, if using the algorithm of [25] we put F under the form $\Pi\sigma$, σ is not fixed in the logarithmic family, depends on the blown up points and degenerates when a generic blown up point approaches the intersection of two curves.

Therefore this article focuses on the problem of finding new normal forms of contracting germs in intermediate cases of surfaces with fixed σ , such that surfaces are minimal or not and intersection matrices are not fixed. Since usual holomorphic objects, curves or foliations, do not fit in global family, it turns out that birational structures could be the adapted notion. Clearly the problem of their unicity raises.

Main results: In section 2, we define large families $\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow B_J$ of marked surfaces with GSS with fixed second Betti number $n = b_2(S)$ which use the same n charts of blowing-ups identified by a subset $J \subset \{0, \dots, n-1\}$. The base admits a stratification by strata over which

the intersection matrix of the n rational curves is fixed. With these fixed charts, we construct explicit global sections of the direct image sheaf of the vertical vector fields $R^1\Phi_{J,\sigma,*}\Theta$ over B_J , which express the dependence on the parameters of the blown up points: $[\theta_i]$ are the infinitesimal deformations along the rational curves and $[\mu_i]$, $i = 0, \dots, n-1$ the infinitesimal deformations transversaly to the rational curves. Surfaces with non trivial global vector fields exist over an analytic set of codimension at least 2 by [9]. In the following theorem we call a “marked surface” a surface with the choice of a rational curve. It fixes the conjugacy class of a contracting germ. Using a result of A. Teleman (see Appendix) we obtain in section 3,

Theorem 1. 1 *Let (S, C_0) be a minimal marked surface containing a GSS of intermediate type, with $n = b_2(S)$. Let $J = I_\infty(C_0)$ and let $\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow B_J$ be the family of surfaces with GSS associated to J and σ . Then, there exists a non empty hypersurface $T_{J,\sigma} \subset B_J$ containing $Z = \{u \in B \mid h^0(S_u, \Theta_u) > 0\}$ such that for $u \in B_J \setminus T_{J,\sigma}$,*

- a) $\{[\theta_u^i], [\mu_u^i] \mid 0 \leq i \leq n-1\}$ is a base of $H^1(S_u, \Theta_u)$,
- b) $\{[\theta_u^i] \mid O_i \text{ is generic}\}$ is a base of the space of infinitesimal logarithmic deformations $H^1(S_u, \Theta_u(-\text{Log } D_u))$, where D_u is the maximal divisor in S_u .

Moreover

- i) If $T_{J,\sigma}$ intersects a stratum $B_{J,M}$ then $T_{J,\sigma} \cap B_{J,M}$ is a hypersurface in $B_{J,M}$,
- ii) $T_{J,\sigma}$ intersect each stratum $B_{J,M}$ such that the corresponding surfaces admit twisted vector fields and $Z \cap B_{J,M} \subset T_{J,\sigma}$,

Corollary 1. 2 *Any marked surface (S, C_0) belongs to a large family $\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow B_J$ and there is a non empty hypersurface $T_{J,\sigma}$ such that over $B_J \setminus T_{J,\sigma}$ this family is versal.*

This answers to the first question and the result is the best possible because $T_{J,\sigma}$ is never empty. What happens on the hypersurface $T_{J,\sigma}$? Is it possible that there is a curve of isomorphic surfaces? Is the canonical image of a stratum $B_{J,M}$ in the Oeljeklaus-Toma coarse moduli space open? Do we obtain all possible surfaces?

In order to be self-contained, we explain in section 4 some results in [25], because Favre normal forms are more convenient for computations. When the dual graph of the curves is a cycle with only one tree, we show that the new normal forms

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^r z_2^s \right)$$

which are composed of l generic blowing-ups, $n-l$ non generic blowing-ups determined by the matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Gl(2, \mathbb{Z}),$$

and if necessary an invertible polynomial mapping tangent to the identity, give all the possible surfaces. In this situation we give explicitly the missing parameter and show that the hypersurface $T_{J,\sigma}$ is a ramification hypersurface, in particular the canonical mapping from a stratum to the Oeljeklaus-Toma coarse moduli space is a ramified covering. More precisely (see section 5) we have

Theorem 1. 3 *Denote $\mathfrak{s} := p + q + l - 1$ and $d := (r + s) - (p + q)$. We choose $a_0 \in \mathbb{C}^*$ and ϵ such that $\epsilon^{r+s-1} = 1$. Then*

A) *If $r + s - 1$ does not divide $l - d$ or $\lambda \neq 1$ there is a bijective polynomial mapping*

$$\begin{aligned} f_{a_0, \epsilon} : \quad \mathbb{C}^{l-1} &\longrightarrow \mathbb{C}^{l-1} \\ a = (a_1, \dots, a_{l-1}) &\longmapsto (b_{p+q+1}(a), \dots, b_{p+q+l-1}(a)) \end{aligned}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} \right)^r z_2^s \right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^s + \sum_{i=p+q}^s b_i z_2^i, z_2^{r+s} \right),$$

where λ depends only on a_0 by 5.51.

B) If $l - d = K(r + s - 1)$ and $\lambda = 1$, there is a bijective polynomial mapping

$$\begin{aligned} f_{a_0, \epsilon} : \quad \mathbb{C}^{l-1} \times \mathbb{C} &\longrightarrow \mathbb{C}^{l-1} \times \mathbb{C} \\ a = (a_1, \dots, a_{l-1}, a_{l+K}) &\longmapsto (b_{p+q+1}(a), \dots, b_{p+q+l-1}(a), c(a)) \end{aligned}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^r z_2^s \right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^s + \sum_{k=p+q}^s b_k z_2^k + c z_2^{\frac{sk(S)}{k(S)-1}}, z_2^{r+s} \right).$$

Corollary 1.4 *Let S in class VII_0^+ containing a GSS. Suppose that the dual graph of the rational curves contains a cycle with only one tree, then S admits a birational structure. In particular if $b_2(S) \leq 3$ all surfaces admit birational structures.*

On the traces $T_{J\sigma} \cap B_{J,M}$ of the hypersurface $T_{J,\sigma}$ on each stratum $B_{J,M}$ the canonical mapping to the Oeljeklaus-Toma coarse moduli space is ramification locus. There is only one missing parameter denoted a_{l+K} . This answers to the questions (2) and (3) in the case of cycles with one tree. The simplest situation for $b_2(S) = 2$ is treated in details at the end of section 4. It is conjectured that the composition of ρ germs of the type of G give all surfaces with ρ trees of rational curves.

This article stems from discussions with Karl Oeljeklaus and Matei Toma at the university of Osnabrück about the case $b_2(S) = 2$, I thank them for their relevant remarks. I thank Andrei Teleman for fruitful discussions in particular to have explained me that thanks to his results (see Corollary 6.72 in Appendix) the cocycles θ_i and μ_i *cannot* be everywhere independent.

2 Surfaces with Global Spherical Shells

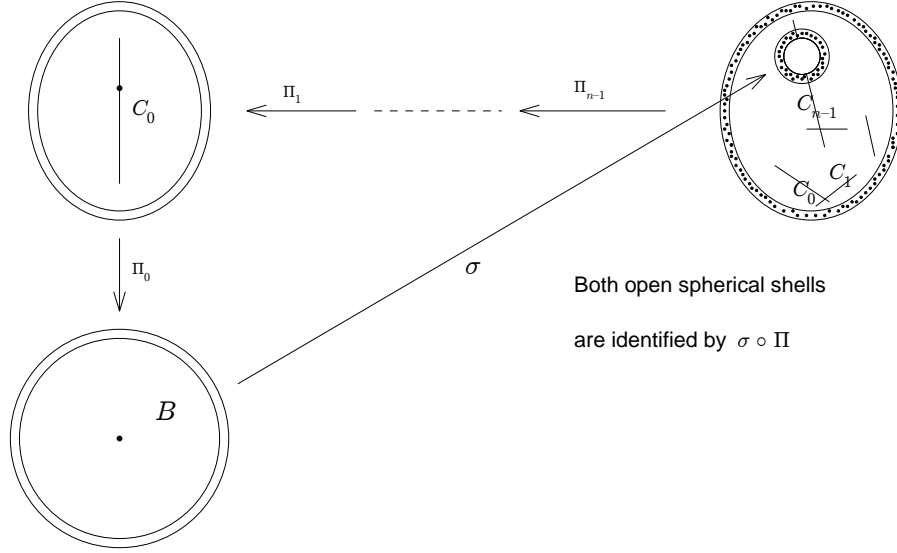
2.1 Basic constructions

Definition 2.5 *Let S be a compact complex surface. We say that S contains a global spherical shell, if there is a biholomorphic map $\varphi : U \rightarrow S$ from a neighbourhood $U \subset \mathbb{C}^2 \setminus \{0\}$ of the sphere S^3 into S such that $S \setminus \varphi(S^3)$ is connected.*

Hopf surfaces are the simplest examples of surfaces with GSS.

Let S be a surface containing a GSS with $n = b_2(S)$. It is known that S contains n rational curves and to each curve it is possible to associate a contracting germ of mapping

$F = \Pi\sigma = \Pi_0 \cdots \Pi_{n-1}\sigma : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ where $\Pi = \Pi_0 \cdots \Pi_{n-1} : B^\Pi \rightarrow B$ is a sequence of n blowing-ups and σ is a germ of isomorphism (see [5]). The surface is obtained by gluing two open shells as explained by the following picture



Definition 2. 6 Let S be a surface containing a GSS, with $n = b_2(S)$. A **Enoki covering** of S is an open covering $\mathcal{U} = (U_i)_{0 \leq i \leq n-1}$ obtained in the following way:

- W_0 is the ball of radius $1 + \epsilon$ blown up at the origin, $C_0 = \Pi_0^{-1}(0)$, $B'_0 \subset\subset B_0$ are small balls centered at $O_0 = (a_0, 0) \in W_0$, $U_0 = W_0 \setminus B'_0$,
- For $1 \leq i \leq n-1$, W_i is the ball B_{i-1} blown up at O_{i-1} , $C_i = \Pi_i^{-1}(O_{i-1})$, $B'_i \subset\subset B_i$ are small balls centered at $O_i \in W_i$, $U_i = W_i \setminus B'_i$.

The pseudoconcave boundary of U_i is patched with the pseudoconvex boundary of U_{i+1} by Π_i , for $i = 0, \dots, n-2$ and the pseudoconcave boundary of U_{n-1} is patched with the pseudoconvex boundary of U_0 by $\sigma\Pi_0$, where

$$\begin{aligned} \sigma : B(1 + \epsilon) &\rightarrow W_{n-1} \\ z = (z_1, z_2) &\mapsto \sigma(z) \end{aligned}$$

is biholomorphic on its image, satisfying $\sigma(0) = O_{n-1}$.

If we want to obtain a minimal surface, the sequence of blowing-ups has to be made in the following way:

- Π_0 blows up the origin of the two dimensional unit ball B ,
- Π_1 blows up a point $O_0 \in C_0 = \Pi_0^{-1}(0), \dots$
- Π_{i+1} blows up a point $O_i \in C_i = \Pi_i^{-1}(O_{i-1})$, for $i = 0, \dots, n-2$, and
- $\sigma : \bar{B} \rightarrow B^\Pi$ sends isomorphically a neighbourhood of \bar{B} onto a small ball in B^Π in such a way that $\sigma(0) \in C_{n-1}$.

Each W_i is covered by two charts with coordinates (u_i, v_i) and (u'_i, v'_i) in which Π_i writes $\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i)$ and $\Pi_i(u'_i, v'_i) = (v'_i + a_{i-1}, u'_i v'_i)$. In these charts the exceptional curves has always the equations $v_i = 0$ and $v'_i = 0$.

A blown up point $O_i \in C_i$ will be called **generic** if it is not at the intersection of two curves. The data (S, C) of a surface S and of a rational curve in S will be called a **marked surface**.

We assume that S is minimal and that we are in the intermediate case, therefore there is at least one blowing-up at a generic point, and one at the intersection of two curves (hence

- Π_1 is a generic blowing-up,
- Π_{n-1} blows-up the intersection of C_{n-2} with another rational curve and
- $\sigma(0)$ is one of the two intersection points of C_{n-1} with the previous curves.

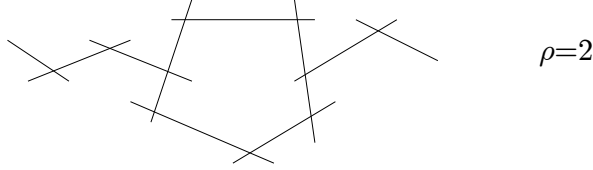
Enoki covering
of a surface with one tree

- $1 \leq l \leq n-1$ and $n \geq 2$. If all, but one, blowing-ups are generic, then $l = n-1$
- For $i = 1, \dots, l-1$, $\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i)$ are generic blowing-ups,
- $\Pi_l(u'_l, v'_l) = (v'_l + a_{l-1}, u'_l v'_l)$ is also generic, but O_l is the origin of the chart (u'_l, v'_l) ,
- For $i = l+1, \dots, n-1$, $\Pi_i(u_i, v_i) = (u_i v_i, v_i)$ or $\Pi_i(u'_i, v'_i) = (v'_i, u'_i v'_i)$ are blowing-ups at the intersection of two curves.

The general case of $\rho \geq 1$ trees is obtained by joining ρ sequences similar to the previous one, i.e.,

$$\begin{aligned}
F &= \Pi\sigma \\
&= (\Pi_0 \cdots \Pi_{l_0-1} \Pi_{l_0} \cdots \Pi_{n_1-1}) \cdots \\
&\quad (\Pi_{n_1+\cdots+n_\kappa} \cdots \Pi_{n_1+\cdots+n_\kappa+l_\kappa-1} \Pi_{n_1+\cdots+n_\kappa+l_\kappa} \cdots \Pi_{n_1+\cdots+n_\kappa+n_{\kappa+1}-1}) \cdots \\
&\quad (\Pi_{n_1+\cdots+n_{\rho-1}} \cdots \Pi_{n_1+\cdots+n_{\rho-1}+l_{\rho-1}-1} \Pi_{n_1+\cdots+n_{\rho-1}+l_{\rho-1}} \cdots \Pi_{n_1+\cdots+n_{\rho-1}-1}) \sigma.
\end{aligned}$$

where $n_1 + \cdots + n_\rho = n$.



We may suppose, up to a conjugation of F by a linear map, that

$$\partial_1 \sigma_2(0) = \frac{\partial \sigma_2}{\partial z_1}(0) = 0$$

it means that the strict transform of the curve $\sigma^{-1}(C_{n-1})$ intersects C_0 at the infinite point of the chart (u, v) , i.e. the origin of (u', v') . This condition is convenient for computations.

When $n = 2$, we denote by $U_{01} = U_0 \cap \Pi_1(U_1) \subset U_0$ and $U_{10} = U_1 \cap \sigma\Pi_0(U_0) \subset U_1$ the two connected components of the intersection $U_0 \cap U_1$ of the images in S of U_0 and U_1 , denoted in the same way.

If $n \geq 3$, $U_{i,i+1} = U_i \cap \Pi_{i+1}(U_{i+1})$, $i = 0, \dots, n-2$, $U_{n-1,0} = U_{n-1} \cap \sigma\Pi_0(U_0)$.

We refer to [5] for the description of configurations of curves. We index the curves $(C_i)_{i \in \mathbb{Z}}$ in the universal covering space following the canonical order (see [5]). Let $a(S) = (a_i)_{i \in \mathbb{Z}}$ be the family of positive integers defined by $a_i = -C_i^2$. By [5] p104, this family is periodic of period n and for any index $i \in \mathbb{Z}$ we define a positive integer independant of i ,

$$2n \leq \sigma_n(S) := \sum_{j=i}^{i+n-1} a_i \leq 3n.$$

The family $(a_i)_{i \in \mathbb{Z}}$ splits into sequences

$$s_p = (p+2, 2, \dots, 2) \quad \text{and} \quad r_m = (2, \dots, 2)$$

of length respectively p and m , where $p \geq 1$ and $m \geq 1$. We call s_p (resp. r_m), $p \geq 1$ ($m \geq 1$) the singular (resp. regular) sequence of length p (resp. m). We have

$$\rho := \#\{\text{trees}\} = \#\{\text{regular sequences}\}.$$

2.2 Large families of marked surfaces

With the previous notations, we consider global families of minimal compact surfaces with the same charts, parameterized by the coordinates of the blown up points on the successive exceptional curves obtained in the construction of the surfaces and such that any marked surface with GSS (S, C_0) belongs to at least one of these families. More precisely, let $F(z) =$

$\Pi_0 \cdots \Pi_{n-1} \sigma(z)$ be a germ associated to any marked surface (S, C_0) with $\text{tr}(S) = 0$. In order to fix the notations we suppose that $C_0 = \Pi_0^{-1}(0)$ meets two other curves (see the picture after definition 2.6), hence $\sigma(0)$ is the intersection of C_{n-1} with another curve. We suppose that

$$\partial_1 \sigma_2(0) = 0.$$

We denote by $I_\infty(C_0) \subset \{0, \dots, n-1\}$ the subset of indices which correspond to blown up points at infinity, that is to say,

$$I_\infty(C_0) := \{i \mid O_i \text{ is the origin of the chart } (u'_i, v'_i)\}.$$

Each generic blow-up

$$\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i) \quad \text{or} \quad \Pi_i(u'_i, v'_i) = (v'_i + a_{i-1}, u'_i v'_i)$$

may be deformed moving the blown up point $(a_{i-1}, 0)$. If we do not want to change the configuration we take

$$\text{for all } \kappa = 0, \dots, \rho-1 \quad (\text{with } n_0 = 0),$$

$$\left\{ \begin{array}{ll} a_{n_1 + \dots + n_\kappa} \in \mathbb{C}^\star, \\ \forall i, 1 \leq i \leq l_\kappa - 1, & a_{n_1 + \dots + n_\kappa + i} \in \mathbb{C}, \\ \forall j, 0 \leq j \leq n_{\kappa+1} - l_\kappa - 1, & a_{n_1 + \dots + n_\kappa + l_\kappa + j} = 0. \end{array} \right.$$

The mapping σ is supposed to be fixed. We obtain a large family of compact surfaces which contains S such that all the surfaces S_a have the same intersection matrix

$$M = M(S_a) = M(S),$$

therefore are logarithmic deformations. For $J = I_\infty(C_0)$ we denote this family

$$\Phi_{J,M,\sigma} : \mathcal{S}_{J,M,\sigma} \rightarrow B_{J,M}$$

where

$$B_{J,M}$$

$$:= \mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \{0\}^{n_1-l_0} \times \dots \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa-1} \times \{0\}^{n_{\kappa+1}-l_\kappa} \times \dots \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1} \times \{0\}^{n_\rho-l_{\rho-1}}$$

$$\simeq \mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \dots \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa-1} \times \dots \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1}$$

and $n_1 + \dots + n_\rho = n$.

In $\mathcal{S}_{J,M,\sigma}$ there is a flat family of divisors $\mathcal{D} \subset \mathcal{S}$ with irreducible components

$$\mathcal{D}_i, \quad i = 0, \dots, n-1,$$

such that for every $a \in B_{J,M}$, $M = (D_{i,a}, D_{j,a})_{0 \leq i, j \leq n-1}$. We may extend this family towards smaller or larger strata which produce minimal surfaces:

- On one hand, **towards a unique Inoue-Hirzebruch surface:** Over

$$\mathbb{C}^{l_0} \times \{0\}^{n_1-l_0} \times \dots \times \mathbb{C}^{l_\kappa} \times \{0\}^{n_{\kappa+1}-l_\kappa} \times \dots \times \mathbb{C}^{l_{\rho-1}} \times \{0\}^{n_\rho-l_{\rho-1}} \simeq \mathbb{C}^{l_0} \times \dots \times \mathbb{C}^{l_\kappa} \times \dots \times \mathbb{C}^{l_{\rho-1}},$$

$$\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow \mathbb{C}^{l_0} \times \dots \times \mathbb{C}^{l_\kappa} \times \dots \times \mathbb{C}^{l_{\rho-1}}.$$

If for an index κ , $a_{n_1+\dots+n_\kappa} = 0$, there is a jump in the configuration of the curves. For instance, if for all κ , $\kappa = 0, \dots, \rho - 1$

$$a_{n_1+\dots+n_\kappa} = \dots = a_{n_1+\dots+n_\kappa+l_\kappa-1} = 0$$

we obtain a Inoue-Hirzebruch surface. To be more precise the base

$$\mathbb{C}^{l_0} \times \dots \times \mathbb{C}^{l_\kappa} \times \dots \times \mathbb{C}^{l_{\rho-1}}$$

splits into locally closed submanifolds called **strata**

- the Zariski open set $\mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \dots \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa-1} \times \dots \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1}$,
- $\rho = C_\rho^1$ codimension one strata

$$\mathbb{C}^\star \times \mathbb{C}^{l_0-1} \times \dots \times \{0\} \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa-2} \times \dots \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1}, \quad 0 \leq \kappa \leq \rho - 1,$$

- $C_{\rho+p-1}^p$ codimension p strata, $1 \leq p := p_0 + \dots + p_{\rho-1} \leq l_0 + \dots + l_{\rho-1}$,

$$\{0\}^{p_0} \times \mathbb{C}^\star \times \mathbb{C}^{l_0-p_0-1} \times \dots \times \{0\}^{p_\kappa} \times \mathbb{C}^\star \times \mathbb{C}^{l_\kappa-p_\kappa-1} \times \dots \times \{0\}^{p_{\rho-1}} \times \mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-p_{\rho-1}-1}$$

- On second hand, **towards Enoki surfaces**. If for all indices such that O_i is at the intersection of two rational curves, in particular for $i \in J$, the blown up point O_i is moved to $O_i = (a_i, 0)$ with $a_i \neq 0$, all the blown up points become generic, the trace of the contracting germ is different from 0. We obtain also all the intermediate configurations.

Proposition 2. 7 *There is a monomial holomorphic function $t : \mathbb{C}^{\text{Card } J} \rightarrow \mathbb{C}$ depending on the variables a_j , $j \in J$ such that over $B_J := \{|t(a)| < 1\} \subset \mathbb{C}^n$, the family $\Phi_{J,\sigma} : S_{J,\sigma} \rightarrow B_J$ may be extended and for every $a \in B_J$, $t(a) = \text{tr}(S_a)$.*

Proof: The trace of a surface does not depend on the germs associated to this surface therefore we may suppose that $O_0 = (a_0, 0)$ is in the chart (u'_0, v'_0) , i.e. $0 \in J$.

Suppose that $\text{Card } J = 1$, then for $i \neq 0$, $\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i)$ and

$$\sigma(z) = (\sigma_1(z) + a_{n-1}, \sigma_2(z)).$$

We have

$$\begin{aligned} F(z) &= \Pi\sigma(z) = \Pi_0(\sigma_1(z)\sigma_2(z)^{n-1} + \sum_{j=0}^{n-1} a_j \sigma_2(z)^j, \sigma_2(z)) \\ (\spadesuit) \quad &= \left(\sigma_2(z), \sigma_1(z)\sigma_2(z)^n + \sum_{j=0}^{n-1} a_j \sigma_2(z)^{j+1} \right), \end{aligned}$$

and with our convention on σ ,

$$\text{tr } DF(0) = \text{tr} \begin{pmatrix} \partial_1 \sigma_2(0) & \partial_2 \sigma_2(0) \\ a_0 \partial_1 \sigma_2(0) & a_0 \partial_2 \sigma_2(0) \end{pmatrix} = \text{tr} \begin{pmatrix} 0 & \partial_2 \sigma_2(0) \\ 0 & a_0 \partial_2 \sigma_2(0) \end{pmatrix} = a_0 \partial_2 \sigma_2(0).$$

The general case is obtained by the composition $F = F_1 \circ \dots \circ F_N$, where $N = \text{Card } J$, F_N of the type of (\spadesuit)

$$F_N(z) = \left(\sigma_2(z), \sigma_1(z)\sigma_2(z)^{m_N} + \sum_{j=0}^{m_N-1} a_j^N \sigma_2(z)^{j+1} \right), \quad m_N \geq 1$$

and other F_k have similar expressions with $\sigma = Id$ and $m_k \geq 1$, i.e.

$$F_k(u, v) = \left(v, uv^{m_k} + \sum_{j=0}^{m_k-1} a_j^k v^{j+1} \right)$$

with $m_1 + \dots + m_N = n$. Therefore

$$F(z) = (\star, \partial_2 \sigma_2(0) a_0^1 a_0^2 \dots a_0^N z_2)$$

and $\text{tr } DF(0) = \partial_2 \sigma_2(0) a_0^1 a_0^2 \dots a_0^N$. \square

Now, $B_J \subset \mathbb{C}^n$ is an open neighbourhood of

$$\mathbb{C}^{l_0} \times \{0\}^{n_1-l_0} \times \dots \times \mathbb{C}^{l_\kappa} \times \{0\}^{n_{\kappa+1}-l_\kappa} \times \dots \times \mathbb{C}^{l_{\rho-1}} \times \{0\}^{n_\rho-l_{\rho-1}}.$$

and we extend the family

$$\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow B_J.$$

thanks to proposition 2.7. We obtain larger strata of minimal surfaces, from dimension $l+1$ to dimension n .

Example 2.8 1) **Example with 2 curves:** For $(3, 2) = -(C_0^2, C_1^2)$, $J = \{1\}$, $O_0 = (a_0, 0)$ and $O_1 = (a_1, 0)$ with $a_0 \in \mathbb{C}^\star$, $a_1 = 0$. The stratum of Inoue-Hirzebruch surface $(4, 2)$ is obtained for $a_0 = 0$, and generic surfaces are obtained for $a_0 \in \mathbb{C}$, $a_1 \neq 0$. If $\sigma(z_1, z_2) = (z_1 + a_1, z_2)$,

$$F(z) = \Pi\sigma(z) = (z_2(z_1 + a_1)(z_2 + a_0), z_2(z_1 + a_1))$$

$\text{tr } DF(0) = a_1$, hence $B_J = \mathbb{C} \times \Delta$.

2) **Example with 6 curves:** If we start with the sequence

$$(42 \ 2 \ 3 \ 3 \ 2) = (s_2 r_1 s_1 s_1 r_1) = -(C_0^2, C_1^2, C_2^2, C_3^2, C_4^2, C_5^2)$$

$J = \{1, 4, 5\}$, and the blown up points are $O_i = (a_i, 0)$, $i = 0, \dots, 5$ with

$$a_0 \in \mathbb{C}^\star, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 \in \mathbb{C}^\star, \quad a_4 = 0, \quad a_5 = 0.$$

Strata towards Inoue-Hirzebruch surfaces are

- $(522 \ 3 \ 3 \ 2)$ when $a_3 = 0$,
- $(42 \ 2 \ 3 \ 42)$, when $a_0 = 0$,
- $(522 \ 3 \ 42)$, when $a_0 = a_3 = 0$, which is a Inoue-Hirzebruch surface with one cycle.

Towards Enoki surfaces, we move each non generic point into generic one:

- $(3 \ 22 \ 3 \ 3 \ 2)$ with $a_1 = 0$, $a_2 \in \mathbb{C}^\star$,
- $(222 \ 3 \ 3 \ 2)$ with $a_1 \in \mathbb{C}^\star$,
- $(42 \ 22 \ 3 \ 2)$ with $a_4 \in \mathbb{C}^\star$,
- $(42 \ 2 \ 3 \ 22)$ with $a_5 \in \mathbb{C}^\star$,
- $(3 \ 222 \ 3 \ 2)$ with $a_1 = 0$, $a_2 \in \mathbb{C}^\star$, $a_4 \in \mathbb{C}^\star$,
- $(2222 \ 3 \ 2)$ with $a_1 \in \mathbb{C}^\star$, $a_4 \in \mathbb{C}^\star$,
- $(3 \ 22 \ 3 \ 22)$ with $a_1 = 0$, $a_2 \in \mathbb{C}^\star$, $a_5 \in \mathbb{C}^\star$,
- $(222 \ 3 \ 22)$ with $a_1 \in \mathbb{C}^\star$, $a_5 \in \mathbb{C}^\star$,
- $(42 \ 2222)$ with $a_4 \in \mathbb{C}^\star$, $a_5 \in \mathbb{C}^\star$,

- (3 22222) with $a_1 = 0$, $a_2 \in \mathbb{C}^*$, $a_4 \in \mathbb{C}^*$, $a_5 \in \mathbb{C}^*$,
- (222222) with $a_1 \in \mathbb{C}^*$, $a_4 \in \mathbb{C}^*$, $a_5 \in \mathbb{C}^*$

Remain non minimal surfaces: we still extend the previous family on a small neighbourhood \widehat{B}_J of B_J , moving the blown up point transversally to the exceptional curves $C_i = \{v_i = 0\} \cup \{v'_i = 0\}$, introducing n new parameters

$$\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i + b_{i-1}), \quad \text{or} \quad \Pi_i(u'_i, v'_i) = (v'_i + a_{i-1}, u'_i v'_i + b_{i-1}), \quad |b_{i-1}| < 1,$$

we obtain

$$\widehat{\Phi}_{J,\sigma} : \widehat{\mathcal{S}}_{J,\sigma} \rightarrow \widehat{B}_J,$$

with $\dim \widehat{B}_J = 2n = 2b_2$. Since for any $(a, b) \in \widehat{B}_J$, $h^1(S_{a,b}, \Theta_{a,b}) = 2b_2(S_{a,b}) + h^0(S_{a,b}, \Theta_{a,b})$, there are some questions:

- Are the parameters a_i, b_i , $i = 0, \dots, n-1$, effective ?
- Which parameter to add when $h^1(S_{a,b}, \Theta_{a,b}) = 2b_2(S_{a,b}) + 1$ in order to obtain a complete family ?
- If we choose $\sigma = Id$ or more generally an invertible polynomial mapping, we obtain a birational polynomial germs. Does this families contain all the isomorphy classes of surfaces with fixed intersection matrix M ?

Remark 2. 9 *It is difficult to determine the maximal domain \widehat{B}_J over which $\widehat{\Phi}_{J,\sigma}$ may be defined. When the surface is minimal, i.e. when $b = (b_0, \dots, b_{n-1}) = 0$, $F_{a,b}(0) = 0$. However, when $b \neq 0$, the fixed point $\zeta = (\zeta_1, \zeta_2)$ moves and the existence condition for the corresponding surface is that the eigenvalues λ_1 and λ_2 of $DF_{a,b}(\zeta)$ satisfy $|\lambda_i| < 1$, $i = 1, 2$.*

2.3 Minimal and non minimal deformations

Let $S = S(F)$ be a minimal surface with GSS and $\mathcal{U} = (U_{i,i+1})$ a Enoki covering of S . We denote by $(e_i)_{0 \leq i \leq n-1}$ the base of the free \mathbb{Z} -module $H_2(S, \mathbb{Z})$ which trivializes the intersection form, i.e. $e_i \cdot e_j = -\delta_{ij}$. Here a **simply minimal divisor** is a connected divisor which may be blown down on a regular point.

Proposition 2. 10 *Let $\widehat{\Phi}_{J,\sigma} : \widehat{\mathcal{S}}_{J,\sigma} \rightarrow \widehat{B}_J$ be a large family of marked surfaces with GSS. Then for any $i = 0, \dots, n-1$ there exists*

- A smooth hypersurface $H_i \subset \widehat{B}_J$,
- A flat family of divisors $\Phi_{J,\sigma} : \mathcal{E}_i \rightarrow \widehat{B}_J \setminus H_i$,

such that

1. For any $(a, b) \in \widehat{B}_J \setminus H_i$, $E_{i,(a,b)}$ is a simply exceptional divisor such that

$$[E_{i,(a,b)}] = e_i,$$

2. $S_{(a,b)}$ contains a simply exceptional divisor $E_{i,(a,b)}$ such that $[E_{i,(a,b)}] = e_i$ if and only if $(a, b) \notin H_i$,
3. Any intersection $H_{i_1} \cap \dots \cap H_{i_p}$ of p different such hypersurfaces is smooth of codimension p .

Proof: The fundamental remark is that $(a, b) \in H_i$ if and only if in the construction of the surface $S_{(a,b)}$ there is a sequence of indices $i, i + j_1, \dots, i + j_p = i \bmod n$ such that the curve C_{i+j_k} is blown up by $C_{i+j_{k+1}}$. If this sequence of blow-ups ends before reaching the index i ,

say at $i + j_q$, $C_i + C_{i+j_1} + \dots + C_{i+j_q}$ would be a simply exceptional divisor. Therefore, the total transform of C_i has to check

$$O_{i-1} = (a_{i-1}, b_{i-1}) \in \Pi_{i-1}^{-1} \dots \Pi_0^{-1} \sigma^{-1} \Pi_{n-1}^{-1} \dots \Pi_{i+1}^{-1}(C_i),$$

or equivalently

$$\Pi_{i+1} \circ \dots \circ \Pi_{n-1} \circ \sigma \circ \Pi_0 \circ \dots \circ \Pi_{i-1}(a_{i-1}, b_{i-1}) \in C_i = \{v_i = 0\}.$$

We have

$$\Pi_{i+1}(u_{i+1}, v_{i+1}) = (u_{i+1}v_{i+1} + a_i, v_{i+1} + b_i) \quad \text{or} \quad \Pi_{i+1}(u'_{i+1}, v'_{i+1}) = (v'_{i+1} + a_i, u'_{i+1}v'_{i+1} + b_i)$$

therefore the condition the equation of H_i is

$$b_i + P(a_0, b_0, \dots, a_{i-1}, b_{i-1}, a_{i+1}, b_{i+1}, \dots, a_{n-1}, b_{n-1}, \widetilde{\sigma}_1(a_{i-1}, b_{i-1}), \widetilde{\sigma}_2(a_{i-1}, b_{i-1})) = 0$$

where

- P is a polynomial,
- $\widetilde{\sigma}(a_{i-1}, b_{i-1}) = \sigma \circ \Pi_0 \dots \circ \Pi_{i-1}(a_{i-1}, b_{i-1})$ does not depend on b_i ,

... and this is the equation of a smooth hypersurface. The third assertion follows readily from the equations. \square

3 Infinitesimal deformations of surfaces with GSS

3.1 Infinitesimal deformations of the families $\mathcal{S}_{J,\sigma}$

We define the following cocycles which are the infinitesimal deformations of the families $\widehat{\mathcal{S}}_{J,\sigma} \rightarrow \widehat{B}_J$:

- For $i = 0, \dots, n-1$, the cocycles θ^i called the “tangent cocycles” move the blown up points O_i along the curve C_i and vanish only (at order two) at the point “at infinity” $C_i \cap C_{i-1}$,
- For $i = 0, \dots, n-1$, the cocycle μ^i called the “transversal cocycles” move O_i transversally to C_i

On a stratum where there are global twisted vector fields we need another infinitesimal deformation it will be defined later.

More precisely,

$$\theta^i = \begin{cases} \frac{\partial}{\partial u_i} & \text{on } U_{i,i+1} \\ 0 & \text{over } U_{j,j+1}, j \neq i \end{cases} \quad \text{If } O_i \text{ belongs to the chart } (u_i, v_i)$$

$$\theta^i = \begin{cases} \frac{\partial}{\partial u'_i} & \text{on } U_{i,i+1} \\ 0 & \text{over } U_{j,j+1}, j \neq i \end{cases} \quad \begin{array}{l} \text{If } O_i \text{ belongs to the chart } (u'_i, v'_i) \\ \text{in particular if } O_i = C_i \cap C_{i-1} \end{array}$$

Since θ^i just moves the blown up point O_i along the curve C_i , all surfaces in these deformations are minimal.

We introduce now n other cocycles which move the blown up point O_i transversally to the

exceptional curves C_i . They yield non minimal surfaces, for instance blown up Hopf surfaces but also surfaces with GSS blown up k times, $1 \leq k \leq n$.
For $i = 0, \dots, n-1$,

$$\mu^i = \begin{cases} \frac{\partial}{\partial v_i} & \text{on } U_{i,i+1} \\ 0 & \text{over } U_{j,j+1}, j \neq i \end{cases} \quad \text{If } O_i \text{ belongs to the chart } (u_i, v_i)$$

$$\mu^i = \begin{cases} \frac{\partial}{\partial v'_i} & \text{on } U_{i,i+1} \\ 0 & \text{over } U_{j,j+1}, j \neq i \end{cases} \quad \begin{array}{l} \text{If } O_i \text{ belongs to the chart } (u'_i, v'_i) \\ \text{in particular if } O_i = C_i \cap C_{i-1} \end{array}$$

For any $J \subset \{0, \dots, n-1\}$, the family $\widehat{\mathcal{S}}_{J,\sigma} \rightarrow \widehat{B}_J$ is globally endowed with a family of Enoki coverings. Using the family of Enoki coverings, all the cocycles θ^i, μ^i are globally defined over $\widehat{\mathcal{S}}_{M,\sigma}$ and give global sections

$$[\theta^i] \in H^0(\widehat{B}_J, R^1 \widehat{\Phi}_{J,\sigma\star} \Theta), \quad [\mu^i] \in H^0(\widehat{B}_J, R^1 \widehat{\Phi}_{J,\sigma\star} \Theta), \quad i = 0, \dots, n-1$$

and more precisely for the l indices i such that O_i is generic

$$[\theta^i] \in H^0(B_{J,M}, R^1 \Phi_{J,M,\sigma\star}(\Theta(-\text{Log } \mathcal{D}))).$$

For any $(a, b) \in \widehat{B}_J$, the cocycles $[\theta^i(a, b)], [\mu^i(a, b)] \in R^1 \Phi_{\star} \Theta_{(a,b)} \otimes \mathbb{C} = H^1(S_{(a,b)}, \Theta_{(a,b)})$, $i = 0, \dots, n-1$ are infinitesimal deformations at $(a, b) \in \widehat{B}_J$ associated to the family $\widehat{\mathcal{S}}_{J,\sigma} \rightarrow \widehat{B}_J$.

3.2 Splitting of the space of infinitesimal deformations

We divide minimal deformation in two types of deformations: logarithmic deformations for which the intersection matrix of the maximal divisor D does not change, in particular the surfaces remain minimal, and deformations in which the cycle may be smoothed at some singular points or disappear and surfaces may become non minimal.

Theorem 3. 11 *Let S be a minimal surface containing a GSS with $b_2(S) = n \geq 1$ rational curves D_0, \dots, D_{n-1} such that $M(S)$ is negative definite. Let U be a spc neighbourhood of D , ρ the number of trees in D , $r_{l_0}, \dots, r_{l_{\rho-1}}$ the corresponding regular sequences and*

$$l = \sum_{i=0}^{\rho-1} l_i$$

the sum of the length of the regular sequences which is also the number of generic blow-ups. Then we have the exact sequence

$$(*) \quad 0 \rightarrow H^1(S, \Theta_S(-\log D)) \rightarrow H^1(S, \Theta_S) \rightarrow H^1(U, \Theta_U) \rightarrow 0.$$

Moreover

$$\dim H^1(S, \Theta(-\log D)) = l + \dim H^0(S, \Theta_S) = 3b_2(S) - \sigma_n(S) + \dim H^0(S, \Theta_S),$$

$$\dim H^1(U, \Theta_U) = 2b_2(S) - l = \sigma_n(S) - b_2(S).$$

Proof: Consider the exact sequence on S

$$(\boxtimes) \quad 0 \rightarrow \Theta_S(-\log D) \rightarrow \Theta_S \rightarrow J_D \rightarrow 0$$

where

$$J_D := \Theta_S / \Theta_S(-\log D) = \bigoplus_{i=0}^{n-1} N_{D_i},$$

$\text{Supp}(J_D) = D$, and N_{D_i} the normal bundle of D_i . The long exact sequence of cohomology gives

$$\cdots \rightarrow H^0(D, J_D) \rightarrow H^1(S, \Theta_S(-\log D)) \rightarrow H^1(S, \Theta_S) \rightarrow H^1(D, J_D) \rightarrow H^2(S, \Theta_S(-\log D)) \rightarrow \cdots$$

If $\theta \in H^0(D, J_D)$ its restriction θ_{D_i} to each curve D_i is a section in the normal bundle N_{D_i} of D_i . Since $D_i^2 \leq -2$, $H^0(D_i, N_{D_i}) = 0$, hence $\theta = 0$ and $H^0(D, J_D) = 0$. Moreover, by [24], thm (1.3), $H^2(S, \Theta_S(-\log D)) = 0$, therefore we have

$$(*) \quad 0 \rightarrow H^1(S, \Theta_S(-\log D)) \rightarrow H^1(S, \Theta_S) \rightarrow H^1(D, J_D) \rightarrow 0.$$

We compute now $H^1(D, J_D)$: the restriction of (\mathbf{X}) to U gives

$$0 \rightarrow H^1(U, \Theta_U(-\log D)) \rightarrow H^1(U, \Theta_U) \rightarrow H^1(D, J_D) \rightarrow 0$$

since by Siu theorem $H^2(U, \Theta_U(-\log D)) = 0$.

Besides, denoting by C the cycle of rational curves and by $H = D - C$ the sum of trees which meet C , we have the exact sequence

$$0 \rightarrow \Theta_U(-\log D) \rightarrow \Theta_U(-\log C) \rightarrow J_H \rightarrow 0$$

where $J_H := \Theta_U(-\log C) / \Theta_U(-\log D)$ and $\text{Supp}(J_H) \subset H$.

By [23] lemma (4.3), $H^1(U, \Theta_U(-\log C)) = 0$, and $H^0(H, J_H) = 0$, hence

$$H^1(U, \Theta_U(-\log D)) = 0$$

With $(*)$ we conclude.

By [3] (see appendix I), $h^1(S, \Theta(-\log D)) = 3b_2(S) - \sigma_n(S) + h^0(S, \Theta)$. Moreover $3b_2(S) - \sigma_n(S)$ is the number of generic blown up points O_i and also is equal to the sum of lengths of regular sequences. \square

3.3 Infinitesimal non logarithmic deformations

We would like to show that $\theta^0, \dots, \theta^{n-1}, \mu^0, \dots, \mu^{n-1}$ are generically linearly independent. We suppose that there exists a linear relation

$$\sum_{i=0}^{n-1} (\alpha_i \theta^i + \beta_i \mu^i) = 0.$$

We choose the curve C_0 such that O_0 is a generic point but O_{n-1} is the intersection of two curves. Hence D_0 the curve in S induced by C_0 is the root of a tree. We shall use this fact later. We have the following linear system where X_i is a vector field over U_i , $i = 0, \dots, n-1$:

$$(E1) \quad \left\{ \begin{array}{lll} X_0 - \Pi_{1\star} X_1 & = & \alpha_0 \frac{\partial}{\partial u_0''} + \beta_0 \frac{\partial}{\partial v_0''} \quad \text{on } U_{01} \subset U_0 \\ \vdots & & \vdots \\ X_i - \Pi_{i+1\star} X_{i+1} & = & \alpha_i \frac{\partial}{\partial u_i''} + \beta_i \frac{\partial}{\partial v_i''} \quad \text{on } U_{i,i+1} \subset U_i \\ \vdots & & \vdots \\ X_{n-2} - \Pi_{n-1\star} X_{n-1} & = & \alpha_{n-2} \frac{\partial}{\partial u_{n-2}''} + \beta_{n-2} \frac{\partial}{\partial v_{n-2}''} \quad \text{on } U_{n-2,n-1} \subset U_{n-2} \\ X_{n-1} - (\sigma \Pi_0)_\star X_0 & = & \alpha_{n-1} \frac{\partial}{\partial u_{n-1}''} + \beta_{n-1} \frac{\partial}{\partial v_{n-1}''} \quad \text{on } U_{n-1,0} \subset U_{n-1} \end{array} \right.$$

where, $u_i'' = u_i$ or $u_i'' = u_i'$ (resp. $v_i'' = v_i$ or $v_i'' = v_i'$).

We notice that by Hartogs theorem, X_i extends to W_i , hence X_i is tangent to C_i for $i = 0, \dots, n-1$; moreover

$$\Pi_{1\star}X_1(O_0) = \dots = \Pi_{n-1\star}X_{n-1}(O_{n-2}) = (\sigma\Pi_0)_\star X_0(O_{n-1}) = 0.$$

Therefore the i -th equation at O_i gives $\beta_i = 0$.

Remark 3. 12 *In fact if we replace the vector field $\frac{\partial}{\partial v_i}$ by any non vanishing transversal vector field, the proof works as well.*

Now, we show that if O_i is the intersection point of two curves, then $\alpha_i = 0$. In fact, there are two cases:

First case $O_i = C_i \cap C_{i-1}$: In the $(i-1)$ -th equation, X_{i-1} and $\frac{\partial}{\partial u_{i-1}}$ or $\frac{\partial}{\partial u_{i-1}'}$ are defined on whole W_{i-1} , therefore it is the same for $\Pi_{i\star}X_i$, so $\Pi_{i\star}X_i$ is tangent to C_{i-1} . As consequence, X_i is tangent to (the strict transform of) C_{i-1} in W_i , thus X_i vanishes at the intersection point $O_i = C_i \cap C_{i-1}$. We have

$$X_i(O_i) = \Pi_{i+1\star}X_{i+1}(O_i) = 0,$$

hence $\alpha_i = 0$.

Second case $O_i = C_i \cap C_k$, $k < i-1$: Then we have $O_{k+1} = C_{k+1} \cap C_k$ and by the previous case,

(1) $\alpha_{k+1} = 0$, therefore

$$X_{k+1} = \Pi_{k+2\star}X_{k+2}.$$

(2) The vector field X_{k+1} is tangent to C_k , therefore X_{k+2} is tangent to (the strict transform of) C_k .

If $O_{k+2} = C_{k+2} \cap C_k$, we have

$$X_{k+2}(O_{k+2}) = \Pi_{k+3\star}X_{k+3}(O_{k+2}) = 0$$

and $\alpha_{k+2} = 0$; by induction we prove $\alpha_{k+1} = \alpha_{k+2} = \dots = 0$ till the moment O_{k+l} is not the point $C_{k+l} \cap C_k$ but the point $C_{k+l} \cap C_{k+l-1}$. However if it happens it means that we are in the first case.

We have obtained

Theorem 3. 13 *The space of non logarithmic infinitesimal deformations $H^1(U, \Theta|_U)$ is generated by the $2b_2(S) - l$ cocycles μ^i , $i = 0, \dots, n-1$ and θ^i for indices i such that O_i is at the intersection of two curves.*

The sequence of blowing-ups splits into subsequences

$$\left(\Pi_{n_1+\dots+n_\kappa} \cdots \Pi_{n_1+\dots+n_\kappa+l_\kappa-1} \right) \circ \left(\Pi_{n_1+\dots+n_\kappa+l_\kappa} \cdots \Pi_{n_1+\dots+n_\kappa+n_{\kappa+1}-1} \right),$$

where $\kappa = 0, \dots, \rho-1$. The indices wich correspond to points O_i at the intersection of two curves are

$$i = n_1 + \dots + n_\kappa + l_\kappa, \dots, n_1 + \dots + n_\kappa + n_{\kappa+1} - 1,$$

therefore for $\kappa = 0, \dots, \rho-1$,

$$\alpha_{n_1+\dots+n_\kappa+l_\kappa} = \dots = \alpha_{n_1+\dots+n_\kappa+n_{\kappa+1}-1} = 0.$$

The equations (E1) become

$$(E2) \quad \left\{ \begin{array}{ll} X_0 - \Pi_{1\star} X_1 & = \alpha_0 \frac{\partial}{\partial u_0} \quad \text{on } W_0 \\ \vdots & \vdots \\ X_{l_0-1} - \Pi_{l_0\star} X_{l_0} & = \alpha_{l_0-1} \frac{\partial}{\partial u_{l_0-1}} \quad \text{on } W_{l_0-1} \\ X_{l_0} - \Pi_{l_0+1\star} X_{l_0+1} & = 0 \quad \text{on } W_{l_0} \\ \vdots & \vdots \\ X_{n_1-1} - \Pi_{n_1\star} X_{n_1} & = 0 \quad \text{on } W_{n_1-1} \\ \vdots & \vdots \\ X_{n_1+\dots+n_\kappa} - \Pi_{n_1+\dots+n_\kappa+1\star} X_{n_1+\dots+n_\kappa+1} & = \alpha_{n_1+\dots+n_\kappa} \frac{\partial}{\partial u_{n_1+\dots+n_\kappa}} \quad \text{on } W_{n_1+\dots+n_\kappa} \\ \vdots & \vdots \\ X_{n_1+\dots+n_\kappa+l_\kappa-1} & \\ -\Pi_{n_1+\dots+n_\kappa+l_\kappa\star} X_{n_1+\dots+n_\kappa+l_\kappa} & = \alpha_{n_1+\dots+n_\kappa+l_\kappa-1} \frac{\partial}{\partial u_{n_1+\dots+n_\kappa+l_\kappa-1}} \quad \text{on } W_{n_1+\dots+n_\kappa+l_\kappa-1} \\ X_{n_1+\dots+n_\kappa+l_\kappa} & \\ -\Pi_{n_1+\dots+n_\kappa+l_\kappa+1\star} X_{n_1+\dots+n_\kappa+l_\kappa+1} & = 0 \quad \text{on } W_{n_1+\dots+n_\kappa+l_\kappa} \\ \vdots & \vdots \\ X_{n_1+\dots+n_{\kappa+1}-1} - \Pi_{n_1+\dots+n_{\kappa+1}\star} X_{n_1+\dots+n_{\kappa+1}} & = 0 \quad \text{on } W_{n_1+\dots+n_{\kappa+1}-1} \\ \vdots & \vdots \\ X_{n_1+\dots+n_{\rho-1}} - \Pi_{n_1+\dots+n_{\rho-1}+1\star} X_{n_1+\dots+n_{\rho-1}+1} & = \alpha_{n_1+\dots+n_{\rho-1}} \frac{\partial}{\partial u_{n_1+\dots+n_{\rho-1}}} \quad \text{on } W_{n_1+\dots+n_{\rho-1}} \\ \vdots & \vdots \\ X_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1} & \\ -\Pi_{n_1+\dots+n_{\rho-1}+l_{\rho-1}\star} X_{n_1+\dots+n_{\rho-1}+l_{\rho-1}} & = \alpha_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1} \frac{\partial}{\partial u_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1}} \quad \text{on } W_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1} \\ X_{n_1+\dots+n_{\rho-1}+l_{\rho-1}} & \\ -\Pi_{n_1+\dots+n_{\rho-1}+l_{\rho-1}+1\star} X_{n_1+\dots+n_{\rho-1}+l_{\rho-1}+1} & = 0 \quad \text{on } W_{n_1+\dots+n_{\rho-1}+l_{\rho-1}} \\ \vdots & \vdots \\ X_{n_1+\dots+n_{\rho-2}} - \Pi_{n_1+\dots+n_{\rho-1}\star} X_{n_1+\dots+n_{\rho-1}} & = 0 \quad \text{on } W_{n_1+\dots+n_{\rho-2}} \\ X_{n-1} - (\sigma\Pi_0)_\star X_0 & = 0 \quad \text{on } W_{n-1} \end{array} \right.$$

It should be noticed that a block may be reduced to one line, if $l_{\kappa} = n_{\kappa+1} - 1$, i.e. if there is in the block only one blowing-up at the intersection of two curves.

For $\kappa = 0, \dots, \rho - 1$, the vector fields $X_{n_1+\dots+n_\kappa+l_\kappa}, \dots, X_{n_1+\dots+n_{\kappa+1}-1}$ glue together into

a vector field that we shall denote $X_{n_1+\dots+n_\kappa+l_\kappa}$. Hence setting

$$\begin{aligned}
\Pi'_0 &= \Pi_0 \cdots \Pi_{l_0-1}, & \Pi''_0 &= \Pi_{l_0} \cdots \Pi_{n_1-1} \\
&\vdots & &\vdots \\
\Pi'_\kappa &= \Pi_{n_1+\dots+n_\kappa} \cdots \Pi_{n_1+\dots+n_\kappa+l_\kappa-1}, & \Pi''_\kappa &= \Pi_{n_1+\dots+n_\kappa+l_\kappa} \cdots \Pi_{n_1+\dots+n_{\kappa+1}-1} \\
&\vdots & &\vdots \\
\Pi'_{\rho-1} &= \Pi_{n_1+\dots+n_{\rho-1}} \cdots \Pi_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1}, & \Pi''_{\rho-1} &= \Pi_{n_1+\dots+n_{\rho-1}+l_{\rho-1}} \cdots \Pi_{n_1+\dots+n_\rho-1}
\end{aligned}$$

$$\Pi = \Pi'_0 \Pi''_0 \cdots \Pi'_\kappa \Pi''_\kappa \cdots \Pi'_{\rho-1} \Pi''_{\rho-1}.$$

we reduce the system to

$$(E3) \quad \left\{ \begin{array}{ll}
X_0 - \Pi_{1\star} X_1 & = \alpha_0 \frac{\partial}{\partial u_0} \quad \text{on } W_0 \\
\vdots & \vdots \\
X_{l_0-2} - \Pi_{l_0-1\star} X_{l_0-1} & = \alpha_{l_0-2} \frac{\partial}{\partial u_{l_0-2}} \quad \text{on } W_{l_0-2} \\
X_{l_0-1} - \Pi''_{0\star} \Pi_{n_1\star} X_{n_1} & = \alpha_{l_0-1} \frac{\partial}{\partial u_{l_0-1}} \\
& \quad \text{on } W_{l_0-1} \cup \dots \cup W_{n_1-1} \\
\vdots & \vdots \\
X_{n_1+\dots+n_\kappa} - \Pi_{n_1+\dots+n_\kappa+1\star} X_{n_1+\dots+n_\kappa+1} & = \alpha_{n_1+\dots+n_\kappa} \frac{\partial}{\partial u_{n_1+\dots+n_\kappa}} \quad \text{on } W_{n_1+\dots+n_\kappa} \\
\vdots & \vdots \\
X_{n_1+\dots+n_\kappa+l_\kappa-2} & \\
-\Pi_{n_1+\dots+n_\kappa+l_\kappa-1\star} X_{n_1+\dots+n_\kappa+l_\kappa-1} & = \alpha_{n_1+\dots+n_\kappa+l_\kappa-2} \frac{\partial}{\partial u_{n_1+\dots+n_\kappa+l_\kappa-2}} \\
& \quad \text{on } W_{n_1+\dots+n_\kappa+l_\kappa-2} \\
X_{n_1+\dots+n_\kappa+l_\kappa-1} & \\
-\Pi''_{\kappa\star} \Pi_{n_1+\dots+n_{\kappa+1}\star} X_{n_1+\dots+n_{\kappa+1}} & = \alpha_{n_1+\dots+n_\kappa+l_\kappa-1} \frac{\partial}{\partial u_{n_1+\dots+n_\kappa+l_\kappa-1-1}} \\
& \quad \text{on } W_{n_1+\dots+n_\kappa+l_\kappa-1} \cup \dots \cup W_{n_1+\dots+n_{\kappa+1}-1} \\
\vdots & \vdots \\
X_{n_1+\dots+n_{\rho-1}} - \Pi_{n_1+\dots+n_{\rho-1}+1\star} X_{n_1+\dots+n_{\rho-1}+1} & = \alpha_{n_1+\dots+n_{\rho-1}} \frac{\partial}{\partial u_{n_1+\dots+n_{\rho-1}}} \\
& \quad \text{on } W_{n_1+\dots+n_{\rho-1}} \\
\vdots & \vdots \\
X_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-2} & \\
-\Pi_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1\star} X_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1} & = \alpha_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-2} \frac{\partial}{\partial u_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-2}} \\
& \quad \text{on } W_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-2} \\
X_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1} - (\Pi''_{\rho-1} \sigma \Pi_0)_\star X_0 & = \alpha_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1} \frac{\partial}{\partial u_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1}} \\
& \quad \text{on } W_{n_1+\dots+n_{\rho-1}+l_{\rho-1}-1} \cup \dots \cup W_{n-1}
\end{array} \right.$$

When $\rho = 1$, i.e. there is only one tree, the linear system reduces to

$$(E4) \quad \begin{cases} X_0 - \Pi_{1*} X_1 & = & \alpha_0 \frac{\partial}{\partial u_0} & \text{over } W_0 \\ \vdots & \vdots & & \\ X_{l-2} - \Pi_{l-1*} X_{l-1} & = & \alpha_{l-2} \frac{\partial}{\partial u_{l-2}} & \text{over } W_{l-2} \\ X_{l-1} - (\Pi'' \circ \sigma \circ \Pi_0)_* X_0 & = & \alpha_{l-1} \frac{\partial}{\partial u_{l-1}} & \text{over } W_{l-1} \cup \dots \cup W_{n-1} \end{cases}$$

Corollary 3. 14 *A relation among the cocycles $[\theta^i]$ and $[\mu^i]$, $i = 0, \dots, n-1$ contains only $[\theta^i]$ in $H^1(S, \Theta(-\text{Log } D))$, i.e. indices for which the blown up point O_i is generic.*

Corollary 3. 15 ([23]) *Let S be a Inoue-Hirzebruch surface with Betti number $b_2(S) = n \geq 1$, then the cocycles θ^i and μ^i , $i = 0, \dots, n-1$ define the versal deformation and the versal logarithmic deformation is trivial. Moreover an Inoue-Hirzebruch surface $S = S_0$ with two cycles of rational curves Γ and Γ' can be deformed into a Hopf surface with two elliptic curves Γ_u and Γ'_u blown up respectively $-\Gamma^2$ and $-\Gamma'^2$ times.*

Proof: In the explicit construction of Inoue-Hirzebruch surfaces [6], there is no generic blown up points and $h^1(S, \Theta) = 2n$, hence we have an explicit base of $H^1(S, \Theta)$ and explicit universal deformation. It is easy to see that any singular point of a cycle may be smoothed for even as well odd Inoue-Hirzebruch surface. \square

For moduli space of Oeljeklaus-Toma see [25] or section 4 below.

Corollary 3. 16 *Fix any J , M , σ and consider a large family $\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow B_J$, then the image of the stratum $B_{J,M}$ in the Oeljeklaus-Toma moduli space of surfaces with intersection matrix M contains an open set.*

Proof: Any large family degenerates to Inoue-Hirzebruch surfaces and at the point $O_{IH} \in B_J$ corresponding to this Inoue-Hirzebruch surface, the family is versal. The point O_{IH} is in the closure of any stratum. By openness of the versality it is versal in a neighbourhood hence on an open set of any stratum. Since the family is generically versal, the image in Oeljeklaus Toma coarse moduli space contains an open set. \square

3.4 Infinitesimal logarithmic deformations

A relation is only possible among infinitesimal logarithmic deformation. In fact it cannot contain θ^i when the curve C_i meets two other curves. In order to be readable and to avoid an overflow of notations, we give a complete proof for surfaces with only one tree and we postpone it to the appendix. The idea of the computation is to work in the first infinitesimal neighbourhood of the maximal divisor. Vanishing of other coefficients should imply to work (if possible) in the successive infinitesimal neighbourhoods.

Proposition 3. 17 *Let (S, C_0) be a marked surface. If $\sum_{i=0}^{n-1} \alpha_i [\theta^i] = 0$ is a relation, then $\alpha_k = 0$ for any index k such that one of the two conditions is fulfilled*

- O_k is the intersection of two rational curves,
- O_k is a generic point but C_k meets two other curves.

In particular, if the unique regular sequences r_m are reduced to one curve (i.e. $m = 1$), $\{[\theta^i], [\mu^i] \mid 0 \leq i \leq n-1\}$ (resp. $\{[\theta^i] \mid O_i \text{ is a generic point}\}$) is an independant family of $H^1(S, \Theta)$ (resp. of $H^1(S, \Theta(-\text{Log } D))$) and a base if there is no non trivial global vector fields.

Remark 3. 18 By induction it is possible to show that for any $k < r + s - (p + q)$, a similar Cramer system may be defined and that $\alpha_k = 0$. However, it is not possible to achieve the proof in this way because when $k = r + s - (p + q)$ a new unknown appears. This difficulty is explained by the fact that in general there is a relation or a class vanishes among the θ^i 's.

3.5 Existence of relations among the tangent cocycles

In this section we show that the cocycles $\{\theta_i \mid O_i \text{ is generic}\}$ cannot be linearly independent everywhere, there exist an obstruction.

Lemma 3. 19 Let S be a minimal surface containing a GSS for which $n = b_2(S) \geq 2$ and $H^0(S, \Theta) \neq 0$ and $\text{tr}(S) = 0$. Let (S, Φ, U) be the versal deformation of $S \simeq S_0$, and

$$Z = \{u \in U \mid h^0(S_u, \Theta_u) > 0\}, \quad M = \bigcap_{i=0}^{n-1} H_i = \{u \in U \mid S_u \text{ is minimal}\}$$

and

$$T = \{u \in U \mid t(u) = 0\},$$

where $t(u) = \text{tr}(S_u)$ is the trace of S_u . Then

- i) $Z \cap (M \setminus T)$ is empty,
- ii) $\text{codim } Z \geq 2$.

Proof: 1) The only minimal surfaces S_u with $t(u) \neq 0$ which admit a non-trivial vector field are Inoue surfaces with an elliptic curve E such that $E^2 = -n$ and for such surfaces we have $K^{-1} = [E + D]$, in particular we have also $h^0(S_u, K^{-1}) \neq 0$.

2) Denote by \mathcal{K} the relative canonical line bundle. Suppose there exists a sequence of points (u_p) in $M \setminus T$ with

$$\lim_{p \rightarrow \infty} u_p = 0 \quad \text{and} \quad h^0(S_{u_p}, \Theta_{u_p}) \neq 0.$$

Then $h^0(S_{u_p}, K_{u_p}^{-1}) \neq 0$ and by Grauert semi-continuity theorem, $h^0(S_0, K_0^{-1}) \neq 0$. Therefore S has, in the same time, non-trivial vector fields and non-trivial sections of $-K$. However, if there are topologically trivial line bundles L^λ and L^κ such that,

$$H^0(S, \Theta \otimes L^\lambda) \neq 0, \quad \text{and} \quad H^0(S, K^{-1} \otimes L^\kappa) \neq 0$$

the relation between λ and κ is by [9], $\lambda = k(S)\kappa$, with $k(S) \geq 2$, therefore it is impossible and we have i).

3) If the configuration of curves allows the existence of twisted vector fields (see [9]),

$$\theta \in H^0(S_u, \Theta_u \otimes L^{\lambda(u)}),$$

the holomorphic function λ is non constant on the logarithmic deformation of S . There is a non-trivial vector field on S_u if and only if $\lambda(u) = 1$, therefore $\text{codim}_{M \cap T} Z \cap M \cap T \geq 1$ in $M \cap T$. With i), it shows that $\text{codim}_M Z \cap M \geq 2$ in M and ii) follows. \square

Lemma 3. 20 Let (S, C_0) be a marked minimal surface containing a GSS of intermediate type with $n = b_2(S)$. Let $(S_{J,\sigma}, \Phi_{J,\sigma}, B_J)$ be a large family of minimal marked surfaces containing S . Then, there exists a non empty hypersurface $T_{J,\sigma}$ such that the family

$$\{\theta_a^i, \mu_a^i \mid 0 \leq i \leq n-1\}$$

is linearly independent in $H^1(S_a, \Theta_a)$ if and only if $a \in B_J \setminus T_{J,\sigma}$.

Proof: Since the family $\{\theta_a^i, \mu_a^i \mid 0 \leq i \leq n-1\}$ is linearly independent in a neighbourhood of the Inoue-Hirzebruch surface, there is at most a hypersurface over which there exist relations. We take $\mathcal{E} = \Theta$ with

$$Z = \{a \in B_{M,\sigma} \mid h^0(S_a, \Theta_a) \neq 0\},$$

which is at least 2-codimensional by lemma 3.19, and we apply the theorem 6.70 of Appendix II \square

Theorem 3. 21 *Let (S, C_0) be a minimal marked surface containing a GSS of intermediate type, with $n = b_2(S)$. Let $J = I_\infty(C_0)$ and let $\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow B_J$ be the family of surfaces with GSS associated to J and σ . Then, there exists a non empty hypersurface $T_{J,\sigma} \subset B$ containing $Z = \{u \in B \mid h^0(S_u, \Theta_u) > 0\}$ such that for $u \in B_J \setminus T_{J,\sigma}$,*

- a) $\{[\theta_u^i], [\mu_u^i] \mid 0 \leq i \leq n-1\}$ is a base of $H^1(S_u, \Theta_u)$,
- b) $\{[\theta_u^i] \mid O_i \text{ is generic}\}$ is a base of $H^1(S_u, \Theta_u(-\text{Log } D_u))$.

Moreover

- i) If $T_{J,\sigma}$ intersects a stratum $B_{J,M}$ then $T_{J,\sigma} \cap B_{J,M}$ is a hypersurface in $B_{J,M}$,
- ii) $T_{J,\sigma}$ intersect each stratum $B_{J,M}$ such that the corresponding surfaces admit twisted vector fields and $Z \cap B_{J,M} \subset T_{J,\sigma}$,

Proof: At the point $a = (a_i)$ where $a_i = 0$ for all i , S_a is a Inoue-Hirzebruch surface. By Corollary 3.15, the family $\{[\theta^i], [\mu^i] \mid 0 \leq i \leq n-1\}$ is a base of $H^1(S_a, \Theta_a)$ therefore has the same property in a neighbourhood. Outside Z , $R^1 \Pi_* \Theta$ is locally free sheaf of rank $2n$, therefore this family is free outside perhaps a hypersurface $T_{J,\sigma}$. At a generic point of $a \in B_J$, S_a is a Enoki surface. If $a_i = 0$ for exactly one index $i \in J$, we have $\sigma_n(S_a) = 2n+1$ and for this configuration of curves there exists twisted vector fields, therefore by lemma 3.20, $T_{J,\sigma}$ is not empty and contains Z . This gives ii).

i) If $T_{J,\sigma}$ which is closed, contains a stratum $B_{J,M}$ it contains smaller strata, in particular the Inoue-Hirzebruch surface, which is impossible. \square

Remark 3. 22 1) *It is possible to prove that $T_{J,\sigma}$ does not intersect those strata near Inoue-Hirzebruch surfaces which have only regular sequences r_1 .*

2) *We shall see that for $\sigma = \text{Id}$, any surfaces with only one tree, $T_{J,\sigma}$ is a ramification locus of $B_{J,\sigma}$ over the Oeljeklaus-Toma moduli space.*

4 Moduli spaces of surfaces with GSS

The goal of this section is to compare the Oeljeklaus-Toma logarithmic families of surfaces with the strata in large families of surfaces $\Phi_{J,M,\sigma} : \mathcal{S}_{J,M,\sigma} \rightarrow B_{J,M}$ which have the same intersection matrix M . In the case of surfaces with only one tree it turns out that we obtain all the surfaces.

4.1 Oeljeklaus-Toma logarithmically versal family

We recall the results of [25] used in the sequel with a small correction described in the remark 4.26.

All surfaces of intermediate type may be obtained from a polynomial germ in the following normal form obtained by [13] and improved by [25].

$$(CG) \quad F(z_1, z_2) = (\lambda z_1 z_2^s + P(z_2) + c z_2^{\frac{s k}{k-1}}, z_2^k)$$

where $k, s \in \mathbb{Z}$, $k > 1$, $s > 0$, $\lambda \in \mathbb{C}^*$,

$$P(z_2) = c_j z_2^j + c_{j+1} z_2^{j+1} + \cdots + c_s z_2^s$$

is a complex polynomial satisfying the conditions

$$0 < j < k, \quad j \leq \mathfrak{s}, \quad c_j = 1, \quad c \in \mathbb{C}, \quad \gcd\{k, m \mid c_m \neq 0\} = 1$$

with $c = 0$ whenever $\frac{\mathfrak{s}k}{k-1} \notin \mathbb{Z}$ or $\lambda \neq 1$.

Lemma 4. 23 ([25], §4) *Two polynomial germs F and*

$$\tilde{F}(z_1, z_2) = \left(\tilde{\lambda} z_1 \tilde{z}_2^{\tilde{\mathfrak{s}}} + \tilde{P}(z_2) + \tilde{c} z_2^{\frac{\tilde{\mathfrak{s}}\tilde{k}}{k-1}}, \quad z_2^{\tilde{k}} \right),$$

in normal form (CG) are conjugated if and only if there exists $\epsilon \in \mathbb{C}$, $\epsilon^{k-1} = 1$ such that

$$\tilde{k} = k, \quad \tilde{\mathfrak{s}} = \mathfrak{s}, \quad \tilde{\lambda} = \epsilon^{\mathfrak{s}} \lambda, \quad \tilde{P}(z_2) = \epsilon^{-j} P(\epsilon z_2), \quad \tilde{c} = \epsilon^{\frac{\mathfrak{s}k}{k-1}} c.$$

Intermediate surfaces admitting a global non-trivial twisted vector field or a non-trivial section of the anticanonical line bundle are exactly those for which $(k-1) \mid \mathfrak{s}$. When moreover $\lambda = 1$ we have a non-trivial global vector field.

Definition 4. 24 *Let S be a surface containing a GSS. The least integer $\mu \geq 1$ such that there exists $\kappa \in \mathbb{C}^*$ for which*

$$H^0(S, K_S^{-\mu} \otimes L^{\kappa}) \neq 0$$

*is called the **index** of S .*

If S is defined by the polynomial germ

$$(CG) \quad F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2) + c z_2^{\frac{\mathfrak{s}k}{k-1}}, \quad z_2^k)$$

then by [25] Remark 4.5,

$$\text{index}(S) := \mu = \frac{k-1}{\gcd(k-1, \mathfrak{s})}.$$

Notice that these germs show the existence of a foliation whose leaves are defined by $\{z_2 = \text{constant}\}$, however *they are not birational*.

The set of polynomial germs

$$F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2), \quad z_2^k)$$

with $c = 0$ are called in pure normal form.

Definition 4. 25 ([25] Def 4.7) *For fixed k and \mathfrak{s} and for a polynomial germ*

$$(CG) \quad F(z_1, z_2) = (\lambda z_1 z_2^{\mathfrak{s}} + P(z_2) + c z_2^{\frac{\mathfrak{s}k}{k-1}}, \quad z_2^k)$$

we define inductively the following finite sequences of integers

$$j =: m_1 < \dots < m_{\rho} \leq \mathfrak{s}, \quad \text{and} \quad k > i_1 > i_2 > \dots > i_{\rho} = 1,$$

by:

$$(i) \quad m_1 := j, \quad i_1 := \gcd(k, m_1),$$

$$(ii) \quad m_{\alpha} := \min\{m > m_{\alpha-1} \mid c_m \neq 0, \gcd(i_{\alpha-1}, m) < i_{\alpha-1}\}, \quad i_{\alpha} = \gcd(k, m_1, \dots, m_{\alpha}) = \gcd(i_{\alpha-1}, m_{\alpha}),$$

$$(iii) \quad 1 = i_{\rho} := \gcd(k, m_1, \dots, m_{\rho-1}, m_{\rho}) < \gcd(k, m_1, \dots, m_{\rho-1}).$$

*We call (m_1, \dots, m_{ρ}) the **type** of F and ρ the **length of the type**. If $\rho = 1$, we say that F is **simple**.*

By [25], §6, the length of the type is exactly the number ρ of trees previously introduced.

Remark 4. 26 1) If the length is $\rho = 1$, then $\gcd(k, j) = 1$ and there is no extra condition on the coefficients $c_{j+1}, \dots, c_{\mathfrak{s}}$, therefore the parameter space of polynomial germs in pure form with integers k, \mathfrak{s} and type j is

$$U_{k, \mathfrak{s}, j} = \mathbb{C}^* \times \mathbb{C}^{\mathfrak{s}-j}.$$

If the length of the type is $\rho \geq 2$, notice that by definition, we have $c_{m_\alpha} \in \mathbb{C}^*$, $\alpha = 1, \dots, \rho$, and $c_{m_1} = c_j = 1$, however between c_{m_α} and $c_{m_{\alpha+1}}$, the coefficients

$$c_{m_\alpha + i_\alpha}, c_{m_\alpha + 2i_\alpha}, \dots, c_{m_\alpha + \left\lceil \frac{m_{\alpha+1} - m_\alpha}{i_\alpha} \right\rceil i_\alpha} \in \mathbb{C}$$

may take any value, but all the other coefficients from $c_{m_\alpha+1}$ to $c_{m_{\alpha+1}-1}$ should vanish. Let

$$\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho) := \sum_{\alpha=1}^{\rho-1} \left\lceil \frac{m_{\alpha+1} - m_\alpha}{i_\alpha} \right\rceil + t - m_\rho$$

then the parameter space of all the germs F with the same integers \mathfrak{s}, k and of the same type (m_1, \dots, m_ρ) in pure form are parameterized by

$$(\mathbb{C}^*)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)}.$$

There exists a family of surfaces

$$\mathcal{S}_{k, \mathfrak{s}, m_1, \dots, m_\rho} \rightarrow (\mathbb{C}^*)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)}$$

such that for every $u \in (\mathbb{C}^*)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)}$, S_u is associated to the germ F_u . We have

Theorem and Definition 4. 27 ([25], thm 7.13) *With the above notations we have:*

- If $k - 1$ does not divide \mathfrak{s} , the family

$$\mathcal{S}_{k, \mathfrak{s}, m_1, \dots, m_\rho} \rightarrow (\mathbb{C}^*)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} =: U_{k, \mathfrak{s}, m_1, \dots, m_\rho}$$

is logarithmically versal at every point and contains all surfaces with parameters \mathfrak{s}, k and type (m_1, \dots, m_ρ) .

- If $k - 1$ divides \mathfrak{s} , the family

$$\mathcal{S}_{k, \mathfrak{s}, m_1, \dots, m_\rho} \rightarrow (\mathbb{C}^*)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \mathbb{C} =: U_{k, \mathfrak{s}, m_1, \dots, m_\rho}$$

- is logarithmically complete at every point,
- is logarithmically versal at every point of

$$U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1} := (\mathbb{C}^*)^{\rho-1} \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \mathbb{C}$$

and its restriction

$$\mathcal{S}_{k, \mathfrak{s}, m_1, \dots, m_\rho} \rightarrow U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1}$$

contains all surfaces with parameters \mathfrak{s}, k and type (m_1, \dots, m_ρ) admitting a non-trivial global vector field,

Moreover its restriction

$$\mathcal{S}_{k, \mathfrak{s}, m_1, \dots, m_\rho} \rightarrow \mathbb{C} \setminus \{0, 1\} \times (\mathbb{C}^*)^{\rho-1} \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} := U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0}$$

is logarithmically versal at every point and contains all surfaces with parameters k, \mathfrak{s} and type (m_1, \dots, m_ρ) without non-trivial global vector fields.

We shall call this family the **Oeljeklaus-Toma logarithmic family of parameters k, \mathfrak{s} and type (m_1, \dots, m_ρ)** .

By lemma (23), for fixed k, \mathfrak{s} and type (m_1, \dots, m_ρ) , $\mathbb{Z}/(k-1)$ acts on the germs in pure normal form. By [25] (7.14),

$$\begin{aligned} & \bullet \mathcal{M}_{k, \mathfrak{s}, m_1, \dots, m_\rho} := U_{k, \mathfrak{s}, m_1, \dots, m_\rho} / (\mathbb{Z}/(k-1)) \quad \text{if } k-1 \text{ does not divide } \mathfrak{s}, \\ & \bullet \begin{cases} \mathcal{M}_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0} := U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0} / (\mathbb{Z}/(k-1)) \\ \mathcal{M}_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1} := U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1} / (\mathbb{Z}/(k-1)), \end{cases} \quad \text{if } k-1 \text{ divides } \mathfrak{s}, \end{aligned}$$

are coarse moduli spaces, moreover the canonical mappings are ramified covering spaces. By lemma 4.23, the ramification set is the union $T_{k, \mathfrak{s}, m_1, \dots, m_\rho}$ (resp. $T_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0}$, $T_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1}$) of hypersurfaces $\{c_i = 0\}$, with $j+1 \leq i \leq \mathfrak{s}$ such that $c_i \in \mathbb{C}$, in particular

$$\begin{aligned} U_{k, \mathfrak{s}, m_1, \dots, m_\rho} \setminus T_{k, \mathfrak{s}, m_1, \dots, m_\rho} &\rightarrow \mathcal{M}_{k, \mathfrak{s}, m_1, \dots, m_\rho} \\ U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0} \setminus T_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0} &\rightarrow \mathcal{M}_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0} \\ U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1} \setminus T_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1} &\rightarrow \mathcal{M}_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1} \end{aligned}$$

are non ramified covering spaces having $k-1$ sheets.

Remark 4. 28 When $k-1$ divides \mathfrak{s} , all the surfaces over the fiber $(\lambda, a, b) \times \mathbb{C}$ with $(\lambda, a, b) \in \mathbb{C} \setminus \{0, 1\} \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)}$ are isomorphic. Moreover

$$U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda=1} / (\mathbb{Z}/(k-1)) \cup U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{\lambda \neq 1, c=0} / (\mathbb{Z}/(k-1))$$

is not separated. In fact, denote by

$$F_{\lambda, c}(z_1, z_2) = (\lambda z_1 z_2^{\frac{\mathfrak{s}}{k-1}} + P(z_2) + c z_2^{\frac{\mathfrak{s}k}{k-1}}, z_2^k).$$

Then any neighbourhood of $F_{1, c}$ with $c \neq 0$ meets any neighbourhood of $F_{1, 0}$ because if $\lambda \neq 1$,

$$F_{\lambda, c} \sim F_{\lambda, 0}.$$

Proposition and Definition 4. 29 If $k-1$ divides \mathfrak{s} , the restriction

$$\mathcal{S}_{k, \mathfrak{s}, m_1, \dots, m_\rho}^0 \rightarrow (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} := U_{k, \mathfrak{s}, m_1, \dots, m_\rho}^{c=0}$$

of the family

$$\mathcal{S}_{k, \mathfrak{s}, m_1, \dots, m_\rho} \rightarrow (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \mathbb{C} := U_{k, \mathfrak{s}, m_1, \dots, m_\rho}$$

will be called the **Oeljeklaus-Toma family of pure surfaces**. It is versal at every point of

$$\mathbb{C} \setminus \{0, 1\} \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)}$$

and effective at every point of

$$\{1\} \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)}.$$

Since the hypersurface $(\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \{0\}$ is invariant under the action of $\mathbb{Z}/(k-1)$ by (23), the projection

$$pr : (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \mathbb{C} \rightarrow (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \{0\}$$

induces a holomorphic mapping

$$p : (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \mathbb{C} / (\mathbb{Z}/(k-1)) \rightarrow (\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k, \mathfrak{s}, m_1, \dots, m_\rho)} \times \{0\} / (\mathbb{Z}/(k-1)).$$

4.2 The generically logarithmically versal family $\mathcal{S}_{J,M,\sigma}$

Notice that by [25]§6, k, t and the integers (m_1, \dots, m_ρ) determine completely the sequence of self-intersections of the rational curves, i.e. the invariant $\sigma_n(S)$ and the intersection matrix $M = M(S)$. We have two families of logarithmic deformations, the first one

$$\mathcal{S}_{J,M,\sigma} \rightarrow B_{J,M} = (\mathbb{C}^\star \times \mathbb{C}^{l_0-1}) \times \dots \times (\mathbb{C}^\star \times \mathbb{C}^{l_\kappa-1}) \times \dots \times (\mathbb{C}^\star \times \mathbb{C}^{l_{\rho-1}-1}),$$

is generically versal by (3.21), the second one

$$\mathcal{S}_{k,\sigma,m_1,\dots,m_\rho} \rightarrow U_{k,\sigma,m_1,\dots,m_\rho} = \mathbb{C}_\lambda^\star \times (\mathbb{C}^\star)^{\rho-1} \times \mathbb{C}^{\epsilon(k,\sigma,m_1,\dots,m_\rho)}$$

is versal at every point, therefore

$$\dim B_{J,M} = \dim(\mathbb{C}^\star)^\rho \times \mathbb{C}^{\epsilon(k,\sigma,m_1,\dots,m_\rho)},$$

$$l = \rho + \epsilon(k,\sigma,m_1,\dots,m_\rho),$$

and the bases are equal up to permutation of the factors.

Lemma 4. 30 *Let $(g_a)_{a \in B_J}$ be a differentiable family of Gauduchon metrics on $\Phi_{J,\sigma} : \mathcal{S}_{J,\sigma} \rightarrow B_J$, ω_a be its associated $(1,1)$ form and let*

$$\deg_{g_a} : H^1(S_a, \mathcal{O}^\star) \rightarrow \mathbb{R}, \quad \deg_{g_a}(L) = \int_{S_a} c_1(L) \wedge \omega_a$$

be the degree of a line bundle. Then there is a non vanishing differentiable negative function

$$C : B_J \rightarrow \mathbb{R}_-^\star$$

such that for any $L^\lambda \in H^1(S_a, \mathbb{C}^\star) \simeq \mathbb{C}^\star$,

$$\deg_{g_a}(L^\lambda) = C(a) \log |\lambda|.$$

Proof: For any $a \in B_J$ the Lie group morphism $\deg_{g_a} : H^1(S_a, \mathbb{C}^\star) \simeq \mathbb{C}^\star \rightarrow \mathbb{R}$ has the form $\deg_{g_a}(L^\lambda) = C \log |\lambda|$ where $C \neq 0$ since this morphism is surjective. Besides the family of Gauduchon metric depends differentiably on $a \in B_J$, therefore $C : B_J \rightarrow \mathbb{R}$ is always positive or always negative. Now, on Enoki surfaces S_a , denote by Γ_a the topologically trivial cycle of rational curves. Then, by Gauduchon theorem [14], [22], and [9]

$$\text{vol}(\Gamma_a) = \deg_{g_a}([\Gamma_a]) = \deg_{g_a}(L^{t(a)}) = C(a) \log |t(a)|$$

where $t(a) = \text{tr}(S_a)$ is the trace of the surface satisfies $0 < |t(a)| < 1$, therefore $C(a) < 0$. Since $C(a) < 0$ when S_a is a Enoki surface, $C(a) < 0$ everywhere. \square

Remark that a numerically \mathbb{Q} -anticanonical divisor D_{-K} on a surface S is a solution of a linear system whose matrix is the intersection matrix $M = M(S)$ of S . Therefore the index is the least integer m such that mD_{-K} is a divisor and this integer is fixed on any logarithmic family $\Phi_{J,M,\sigma} : \mathcal{S}_{J,M,\sigma} \rightarrow B_{J,M}$.

Lemma 4. 31 *Let $F(z_1, z_2) = (\lambda z_1 z_2^5 + P(z_2) + c z_2^{\frac{sk}{k-1}}, z_2^k)$ be a Favre contracting germ associated to a surface of intermediate type S . Let $\mu = \text{index}(S) \in \mathbb{N}^\star$ and $\kappa \in \mathbb{C}^\star$ such that $H^0(S, K_S^{\otimes -\mu} \otimes L^\kappa) \neq 0$. Then*

$$\kappa = k(S)^{-\mu} \lambda^{-\mu}.$$

Proof: A global section $\theta \in H^0(S, K^{-\mu} \otimes L^\kappa)$ induces a germ $\theta = z_2^\alpha A(z) \left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \right)^{\otimes \mu}$ which satisfies the condition

$$\theta(F(z)) = \kappa \left(\det DF(z) \right)^\mu \theta(z),$$

where α is the vanishing order of θ along C_{n-1} and $A(0) \neq 0$. Since $\det DF(z) = \lambda k z_2^{s+k-1}$, comparison of lower degree terms gives

$$z^{k\alpha} A(0) = \kappa (\lambda k)^\mu z_2^{\mu(s+k-1)+\alpha} A(0)$$

hence

$$\begin{cases} \alpha(k-1) = \mu(k-1+s) \\ \kappa = k(S)^{-\mu} \lambda^{-\mu}. \end{cases}$$

□

Lemma 4. 32 *Let S be a minimal complex surface, μ the index of S and κ such that $H^0(S, K^{\otimes -\mu} \otimes L^\kappa) \neq 0$. Then a section of $K^{\otimes -\mu} \otimes L^\kappa$ vanishes on all the rational curves in S .*

Proof: Let D_i , $i = 0, \dots, n-1$ be the n rational curves in S and suppose that

$$K^{\otimes -\mu} \otimes L^\kappa = \sum_{i=0}^{n-1} k_i D_i.$$

We have $k_i \geq 0$ for all $i = 0, \dots, n-1$; if one coefficient vanishes, say $k_0 = 0$, on one hand, since the maximal divisor is connected,

$$c_1(K^{\otimes -\mu} \otimes L^\kappa).D_0 = \sum_{i=1}^{n-1} k_i D_i D_0 > 0$$

and on second hand, by adjunction formula

$$c_1(K^{\otimes -\mu} \otimes L^\kappa).D_0 = -\mu c_1(K).D_0 = \mu(D_0^2 + 2) \leq 0$$

we obtain a contradiction. □

Proposition 4. 33 *Let $\Phi_{J,M,\sigma} : \mathcal{S}_{J,M,\sigma} \rightarrow B_{J,M}$ be a logarithmic family of marked intermediate surfaces with $J \neq \emptyset$ and μ the common index of the surfaces. Then*

1) *there exists a unique surjective holomorphic function*

$$\begin{array}{ccc} \kappa = \kappa_{J,M,\sigma} : & B_{J,M} & \rightarrow \mathbb{C}^* \\ & a & \mapsto \kappa(a) \end{array}$$

such that $H^0(S_a, K_{S_a}^{-\mu} \otimes L^{\kappa(a)}) \neq 0$.

2) *If the surfaces admit twisted vector fields there exists a unique surjective holomorphic function*

$$\begin{array}{ccc} \lambda = \lambda_{J,M,\sigma} : & B_{J,M} & \rightarrow \mathbb{C}^* \\ & a & \mapsto \lambda(a) \end{array}$$

such that the marked surface $(S_a, C_{0,a})$ is defined by a germ of the form

$$F_a(z_1, z_2) = (\lambda(a) z_1 z_2^s + P_a(z_2), z_2^k).$$

3) *The fibers $K_\alpha := \{\kappa = \alpha\}$ (resp. $\Lambda_\alpha := \{\lambda = \alpha\}$), $\alpha \in \mathbb{C}^*$, are closed in B_J hence analytic in $B_J \supset B_{J,M}$.*

4) *Let $\overline{B_{J,M}} \subset B_J$ be the closure of $B_{J,M}$ in B_J , i.e. the union of $B_{J,M}$ with the smaller strata, then $\kappa_{J,M,\sigma}$ extends holomorphically to $\kappa_{J,\sigma} : \overline{B_{J,M}} \rightarrow \mathbb{C}$ and $\kappa_{J,\sigma}^{-1}(0) = \overline{B_{J,M}} \setminus B_{J,M}$.*

Proof: 1) For $a \in B_{J,M}$, the complex number $\kappa(a)$ satisfies $h^0(S_a, K_{S_a}^{-\mu} \otimes L^{\kappa(a)}) = 1$. It is unique because there is no topologically trivial divisor. We consider a new base space $B_{J,M} \times \mathbb{C}^*$, and let

$$pr_1 : B_{J,M} \times \mathbb{C}^* \rightarrow B_{J,M},$$

be the first projection. Let $\mathcal{K} \rightarrow \mathcal{S}_{J,M,\sigma}$ be the relative canonical line bundle and $\mathcal{L} \rightarrow \mathcal{S}_{J,M,\sigma} \times \mathbb{C}^*$ be the tautological line bundle such that $\mathcal{L}_{a,\tau}$ is the line bundle L^τ over S_a . We consider the family of rank one vector bundles

$$pr_1^* \mathcal{K} \otimes \mathcal{L} \rightarrow pr_1^* \mathcal{S}_{J,M,\sigma} \xrightarrow{pr_1^* \Phi_{J,M,\sigma}} B_{J,M} \times \mathbb{C}^*.$$

Then $(pr_1^* \mathcal{K} \otimes \mathcal{L})_{(a,\alpha)} = K_a \otimes L^\alpha$. The set of points

$$Z = \{(a, \alpha) \in B_{J,M} \times \mathbb{C}^* \mid h^0(S_a, K_a \otimes L^\alpha) > 0\}$$

is an analytic subset. Let

$$pr : Z \rightarrow B_{J,M}$$

be the restriction to Z of the first projection pr_1 over $B_{J,M}$. Then pr is surjective by hypothesis. Each fiber contains only one point. Moreover pr is proper: in fact we consider the closure $\overline{Z} \subset B_{J,M} \times \mathbb{P}^1(\mathbb{C})$. By Remmert-Stein theorem, either \overline{Z} is an analytic set in $B_{J,M} \times \mathbb{P}^1(\mathbb{C})$ or contains at least one of the hypersurfaces $B_{J,M} \times \{0\}$ or $B_{J,M} \times \{\infty\}$. But it is impossible because each fiber contains only one point. Therefore \overline{Z} is analytic and $\overline{pr} : \overline{Z} \rightarrow B_{J,M}$ is proper hence a ramified covering. Since there is only one sheet, it is the graph of a holomorphic mapping $\kappa : B_{J,M} \rightarrow \mathbb{P}^1(\mathbb{C})$. Since for every $a \in B_{J,M}$, $pr^{-1}(a)$ contains exactly one point in \mathbb{C}^* , κ has only values in \mathbb{C}^* .

Now, κ cannot be constant because $\kappa = (k\lambda)^{-\mu}$ and λ is a parameter of a logarithmic versal family, therefore the non-constant mapping $\kappa : (\mathbb{C}^*)^\rho \times \mathbb{C}^{l-\rho} \rightarrow \mathbb{C}^*$ is surjective.

2) By lemma 4.31, $\kappa = k^{-1}\lambda^{-1}$.

3) Consider the hypersurface $\{\kappa = \alpha\} \subset B_{J,M}$ for $\alpha \in \mathbb{C}^*$. The closure $\overline{B_{J,M}}$ of $B_{J,M}$ in B_J is the union of $B_{J,M}$ with lower strata, hence $\overline{B_{J,M}} \setminus B_{J,M}$ is also a hypersurface. Remmert-Stein theorem shows that $\{\kappa = \alpha\}$ is analytic or contains an irreducible component of $\overline{B_{J,M}} \setminus B_{J,M}$. However the second possibility is excluded by Grauert semi-continuity theorem because on a whole stratum we would have $H^0(S_a, K^{-\mu} \otimes L^\alpha) \neq 0$ which is impossible because the twisting parameter is not constant. Therefore the slice has an extension. If $\{\kappa = \alpha\} \cap (\overline{B_{J,M}} \setminus B_{J,M}) \neq \emptyset$, the line bundle $K^{-\mu} \otimes L^\alpha$ has a section over $\mathcal{S}_{J,\sigma|\overline{\{\kappa=\alpha\}}}$ hence the zero locus which is the union of all the rational curves by [9] would be is a flat family of divisors; however it is impossible because the configuration changes contradicting flatness (it can be seen that the curve whose self-intersection decreases has a volume which tends to infinity (see [12])). Therefore $\overline{\{\kappa = \alpha\}} \cap (B_J \setminus B_{J,M}) = \emptyset$ and each slice is already closed in $B_{J,M}$.

4) Since the fibers K_α are closed in B_J ,

$$\lim_{a \rightarrow \overline{B_{J,M}} \setminus B_{J,M}} \kappa_{J,M,\sigma}(a) = 0 \text{ or } \infty.$$

Let \mathcal{K} be the relative canonical line bundle, $\theta \in H^0(\mathcal{S}_{J,M,\sigma}, \mathcal{K}^{-\mu} \otimes L^\kappa)$ be the flat family of sections over $B_{J,M}$ and Z the associated divisor of zeroes of θ . By lemma 4.30,

$$vol(Z_a) = \deg_{g_a}([Z_a]) = \deg_{g_a}(K_a^{-\mu} \otimes L^{\kappa(a)}) = -\mu \deg_{g_a}(K_a) + C(a) \log |\kappa(a)|$$

Since $a \mapsto \deg_{g_a}(K_a)$ is differentiable, hence bounded, and $vol(Z_a) > 0$, the limit of $\kappa = \kappa_{J,M,\sigma}(a)$ near $\overline{B_{J,M}} \setminus B_{J,M}$ cannot be ∞ , therefore κ extends continuously and holomorphically. \square

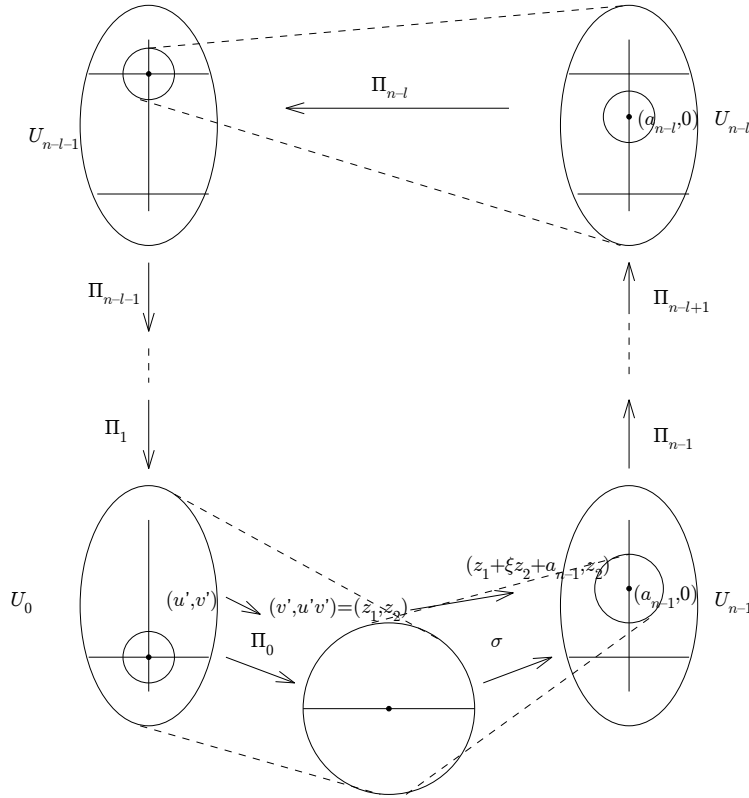
Remark 4. 34 1) If $\text{index}(S) = 1$, we have $\lambda^{-1} = k(S)\kappa$, i.e. the invariant used here is the inverse of the invariant $\lambda = \lambda(S)$ in [9].

2) If $\text{index}(S) \neq 1$, $\lambda(a)$ is defined up to a $(k-1)$ -root of unity.

Lemma 4. 35 Let S be a surface with GSS such that the dual graph of the curves admits exactly one tree. We choose the numbering such that the first blowing-ups are non generic and the last $l \geq 1$ ones generic. More precisely

- $O_0 \in C_0$ is the origin of the chart (u'_0, v'_0) , hence $\Pi_0(u'_0, v'_0) = (v'_0, u'_0 v'_0)$
- $O_i \in C_i$ is the origin of the chart (u_i, v_i) or (u'_i, v'_i) for $i = 1, \dots, n-l-1$,
- $O_{n-l} = (a_{n-l}, 0) \in C_{n-l}$ with $a_{n-l} \in \mathbb{C}^*$, $\Pi_{n-l}(u_{n-l}, v_{n-l}) = (u_{n-l}v_{n-l}, v_{n-l})$,
- $O_i = (a_i, 0) \in C_i$, $i = n-l+1, \dots, n-1$ is in the chart (u_i, v_i) and

$$\Pi_i(u_i, v_i) = (u_i v_i + a_{i-1}, v_i).$$



We suppose that with this choice (the induced curve by C_{n-l} is the root of the tree), σ is a polynomial isomorphism of the special form

$$\sigma(z_1, z_2) = (\sigma_1(z) + a_{n-1}, \sigma_2(z)) = (z_1 + \xi z_2^u + a_{n-1}, z_2), \quad u \geq 1.$$

Let $\mu = \text{index}(X)$ be the index of S . Then on the corresponding base $B_{J,M}$ of the family $\Phi_{J,M,\sigma} : S_{J,M,\sigma} \rightarrow B_{J,M}$, the holomorphic function

$$\kappa = \kappa_{J,M,\sigma} : \mathbb{C}^* \rightarrow \mathbb{C}^*$$

such that $H^0(S_a, K_{S_a}^{-\mu} \otimes L^{\kappa(a)}) \neq 0$ is a monomial holomorphic function of a_{n-l} , where $O_{n-l} = (a_{n-l}, 0)$. More precisely, if $\delta = ps - qr$ and $\sigma = p + q + l - 1$,

$$\kappa_{J,M,\sigma}(a_{n-l}) = \delta^\mu a_{n-l}^{\mu \left[\frac{r\sigma}{r+s-1} - p + 1 \right]}, \quad \mu \left[\frac{r\sigma}{r+s-1} - p + 1 \right] \in \mathbb{N}^*.$$

Proof: Let $\kappa := \kappa_{J,M,\sigma}$ be the holomorphic function given by the proposition 4.33. We have

$$\begin{aligned}\Pi_{n-l} \cdots \Pi_{n-1} \sigma(z) &= \left((z_1 + \xi z_2^u + a_{n-1}) z_2^l + \sum_{i=n-l}^{n-2} a_i z_2^{l-n+i+1}, z_2 \right) \\ &= \left(z_1 z_2^l + \xi z_2^{l+u} + \sum_{i=n-l}^{n-1} a_i z_2^{l-n+i+1}, z_2 \right) \\ \Pi_0 \cdots \Pi_{n-l-1}(u'', v'') &= (u''^p v''^q, u''^r v''^s),\end{aligned}$$

with

$$(u'', v'') = (u_{n-l-1}, v_{n-l-1}) \quad \text{or} \quad (u'', v'') = (u'_{n-l-1}, v'_{n-l-1}), \quad p \leq r, \quad q \leq s, \quad p+q < r+s.$$

Combining these two expressions and using the special form of σ , the expression of F is

$$F(z) = \Pi \sigma = \left(\left(z_1 z_2^l + \xi z_2^{l+u} + \sum_{i=n-l}^{n-1} a_i z_2^{l-n+i+1} \right)^p z_2^q, \left(z_1 z_2^l + \xi z_2^{l+u} + \sum_{i=n-l}^{n-1} a_i z_2^{l-n+i+1} \right)^r z_2^s \right),$$

with $a_{n-l} \neq 0$.

Setting

$$\left\{ \begin{array}{l} \left(\right) := \left(z_1 z_2^l + \xi z_2^{l+u} + \sum_{i=n-l}^{n-1} a_i z_2^{l-n+i+1} \right) \quad \text{and} \\ \left[\right] := \frac{\partial}{\partial z_2} \left(\right) = l z_1 z_2^{l-1} + \xi(l+u) z_2^{l+u-1} + \sum_{i=n-l}^{n-1} a_i (l-n+i+1) z_2^{l-n+i} \end{array} \right.$$

$$DF(z) = \begin{pmatrix} p \left(\right)^{p-1} z_2^{l+q} & p \left(\right)^{p-1} \left[\right] z_2^q + \left(\right)^p q z_2^{q-1} \\ r \left(\right)^{r-1} z_2^{l+s} & r \left(\right)^{r-1} \left[\right] z_2^s + \left(\right)^r s z_2^{s-1} \end{pmatrix}$$

and

$$\det DF(z) = (ps - qr) \left(\right)^{p+r-1} z_2^{l+q+s-1}.$$

Let $\theta \in H^0(S, K_S^{-\mu} \otimes L^\kappa)$, then there exists an invariant germ in a neighbourhood of the origin of the ball still denoted by θ which vanishes on the curves

$$\theta(z) = z_2^\alpha A(z) \left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \right)^{\otimes \mu}$$

such that $A(0) \neq 0$. This germ satisfies the condition

$$\theta(F(z)) = \kappa (\det DF(z))^\mu \theta(z),$$

which is equivalent to

$$\left(\right)^{\alpha r} z_2^{\alpha s} A(F(z)) = \kappa (ps - qr)^\mu \left(\right)^{\mu(p+r-1)} z_2^{\mu(l+q+s-1)+\alpha} A(z)$$

where $\delta := ps - qr = \pm 1$. Considering the homogeneous part of lower degree of each member, we obtain

$$(1) \quad \alpha(r+s-1) = \mu(p+q+l-1+r+s-1) = \mu(\sigma+r+s-1)$$

$$(2). \quad \kappa = (ps - qr)^\mu a_{n-l}^{\alpha r - \mu(p+r-1)}$$

By proposition 4.8 3), κ vanishes on smaller strata, therefore $\alpha r - \mu(p+r-1) > 0$. We derive the value of κ from (1) and (2). \square

Remark that any $F = \Pi\sigma$ with ρ trees splits into ρ contracting germs

$$F = (\Pi_0 \cdots \Pi_{i_1-1})(\Pi_{i_1} \cdots \Pi_{i_2-1}) \cdots (\Pi_{i_{\rho-1}} \cdots \Pi_{n-1}\sigma)$$

If we decompose trivially Π_{i_j} into $\sigma_j \Pi'_{i_j}$, $j = 1, \dots, \rho-1$, $\sigma_\rho = \sigma$ with

$$\Pi'_{i_j}(u'_{i_j}, v'_{i_j}) = (v'_{i_j}, u'_{i_j} v'_{i_j}) = (z_1, z_2), \quad \sigma_j(z_1, z_2) = (z_1, z_2) = (u_{i_j-1}, v_{i_j-1})$$

we have the decomposition

$$F = F_0 \circ \cdots \circ F_j \circ \cdots \circ F_{\rho-1},$$

with $F_j := \Pi'_{i_j} \Pi_{i_j+1} \cdots \Pi_{i_{j+1}-1} \sigma_{j+1}$, such that each germ F_j is the germ of a marked surface S_j with one tree which satisfies the conditions of lemma 4.35. Notice that there is a unique index k_j , $i_j + 1 \leq k_j \leq i_{j+1} - 1$ such that C_{k_j} meets two other curves.

Proposition 4.36 *Let $F = \Pi\sigma = F_0 \circ \cdots \circ F_{\rho-1}$ where each F_i satisfies the conditions of lemma 4.35. Let $\mu = \text{index}(S) \in \mathbb{N}^*$ and $\kappa \in \mathbb{C}^*$ such that $H^0(S, K_S^{-\mu} \otimes L^\kappa) \neq 0$. Then κ depends only on the coordinates $a_i \in \mathbb{C}^*$ of the generic blown up points $O_i = (a_i, 0) \in C_i$ such that C_i meets two other curves (i.e. is the root of a tree). More precisely there is a surjective monomial function of the variables $a_{k_j} \in \mathbb{C}^*$, $j = 0, \dots, \rho-1$*

$$\kappa = \kappa_{J,M,\sigma} : B_{J,M} \rightarrow \mathbb{C}^*$$

such that

- For $a \in B_{J,M}$,

$$H^0(S_a, K_{S_a}^{-\mu} \otimes L^{\kappa_{J,M,\sigma}(a)}) \neq 0$$

- $\kappa_{J,M,\sigma}$ extends holomorphically to $\kappa_{J,\sigma} : \overline{B_{J,M}} \rightarrow \mathbb{C}$ such $\kappa_{J,\sigma}^{-1}(0) = \overline{B_{J,M}} \setminus B_{J,M}$.

Proof: The integer μ depends only on the intersection matrix M , hence is constant on $B_{J,M}$. The complex number κ depends only on the isomorphism class of the surface, hence does not depend on the choice of the germ. Let F be the chosen germ and G a Favre germ in the conjugation class of F . The germ $G(z) = (\lambda z_1 z_2^m + \cdots, z_2^k)$ has a unique Oeljeklaus-Toma decomposition (see [25] Prop. 5.10)

$$G = G_0 \circ \cdots \circ G_j \circ \cdots \circ G_{\rho-1}, \quad G_j(z) = (\lambda_j z_1 z_2^{m_j} + \cdots, z_2^{k_j}).$$

The invariants k and λ are multiplicative, therefore

$$k = k(S) = \prod_{j=0}^{\rho-1} k_j, \quad \lambda = \prod_{j=0}^{\rho-1} \lambda_j,$$

By lemma 4.35,

$$\kappa_j = \delta_j^{\mu_j} a_{k_j}^{\frac{\mu_j(r_j l_j + p_j + s_j - \delta_j - 1)}{r_j + s_j - 1}}, \quad \text{with} \quad \frac{\mu_j(r_j l_j + p_j + s_j - \delta_j - 1)}{r_j + s_j - 1} \in \mathbb{N}^*$$

and by lemma 4.31,

$$\kappa_j = k_j^{-\mu_j} \lambda_j^{-\mu_j}.$$

$$\kappa = k(S)^{-\mu} \lambda^{-\mu} = \left(\prod_{j=0}^{\rho-1} k_j \right)^{-\mu} \left(\prod_{j=0}^{\rho-1} \lambda_j \right)^{-\mu} = \left(\prod_{j=0}^{\rho-1} k_j \lambda_j \right)^{-\mu}.$$

Setting $\mu' = \prod_{j=0}^{\rho-1} \mu_j$, we have,

$$\kappa^{\mu'} = \prod_{j=0}^{\rho-1} \left(\kappa_j^{\mu'/\mu_j} \right)^{\mu} = \prod_{j=0}^{\rho-1} \left(\delta_j^{\mu'/\mu_j} a_{k_j}^{\frac{\mu'(r_j l_j + p_j + s_j - \delta_j - 1)}{r_j + s_j - 1}} \right)^{\mu}.$$

is a monomial function of the variables a_{k_j} therefore $(\kappa^{\mu'})^{-1}(0) = \overline{B_{J,M}} \setminus B_{J,M}$. Moreover

$$\lim_{a \rightarrow \overline{B_{J,M}} \setminus B_{J,M}} \kappa(a) = 0,$$

We conclude by Riemann theorem that κ is a holomorphic function on $\overline{B_{J,M}}$. A power is monomial, therefore κ is itself monomial and

$$\frac{\mu(r_j l_j + p_j + s_j - \delta_j - 1)}{r_j + s_j - 1} \in \mathbb{N}^*.$$

□

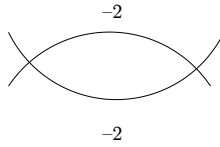
4.3 Surfaces with $b_2 = 2$.

4.3.1 Rational curves

Up to a circular permutation intersection matrix and configuration of the curves D_0 and D_1 are the following:

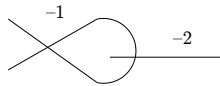
- Surfaces of trace $t \neq 0$: Enoki surfaces and Inoue surfaces,

$$M(S) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}, \quad [D_0] = e_0 - e_1, \quad [D_1] = e_1 - e_0, \quad [D_0] + [D_1] = 0.$$



- Intermediate surface

$$M(S) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad [D_0] = e_0 - e_1 - e_0 = -e_1, \quad [D_1] = e_1 - e_0.$$



- Inoue-Hirzebruch surfaces

$$M(S) = \begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix}, \quad [D_0] = e_0 - e_1 - e_0 - e_1 = -2e_1, \quad [D_1] = e_1 - e_0, \quad [D_0] + [D_1] = -e_0,$$

$$M(S) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [D_0] = e_0 - e_1 - e_0 = -e_1, \quad [D_1] = e_1 - e_0 - e_1 = -e_0.$$



4.3.2 Intermediate surfaces

We consider intermediate surfaces S , since the problem of normal forms is solved for the other cases. There are two curves: one rational curve with a double point $D_1^2 = -1$ with one tree $D_0^2 = -2$, $D_0 D_1 = 1$. Favre polynomial germs are

$$F_c(z_1, z_2) = (\lambda z_1 z_2 + z_2 + c z_2^2, z_2^2)$$

where $k = k(S) = 2$, $\mathfrak{s} = 1$. Invariant vector fields θ exist if and only if $\lambda = 1$ in which case

$$\theta(z) = \alpha z_2^{\mathfrak{s}/(k-1)} \frac{\partial}{\partial z_1} = \alpha z_2 \frac{\partial}{\partial z_1}, \quad \alpha \in \mathbb{C}$$

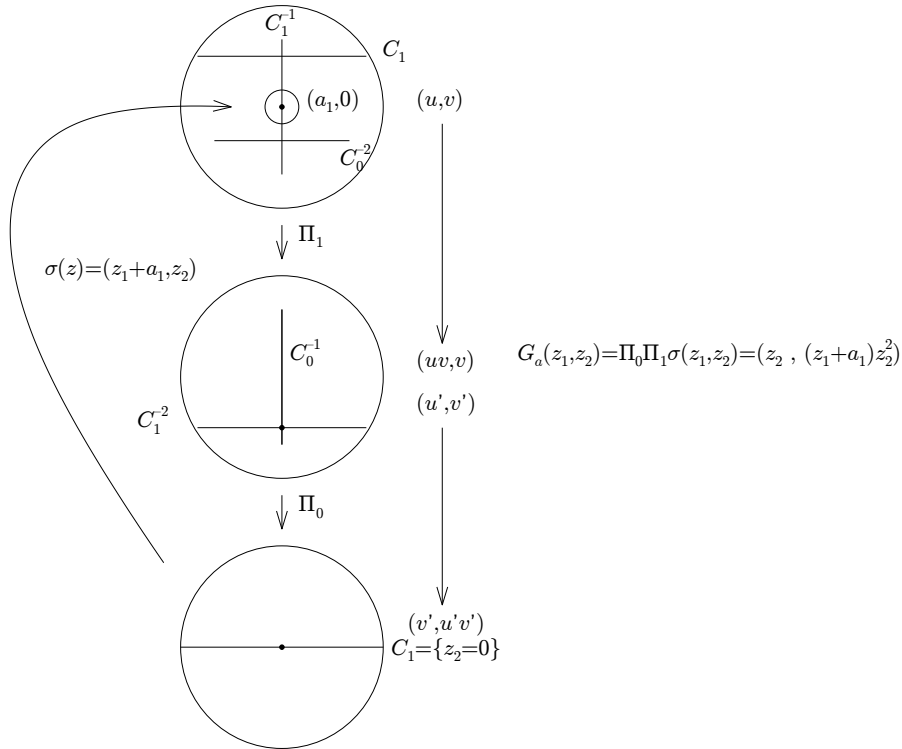
Intermediate surfaces belong to three families, namely for $J = \{0\}$, $J = \{1\}$ and $J = \{0, 1\}$.

Case $J = \{0\}$

The case $J = \{1\}$ is similar.

The family of germs defining surfaces of $\Phi_{J,M,\sigma} : \mathcal{S}_{J,M,\sigma} \rightarrow B_{J,M}$ are

$$G_a^J(z_1, z_2) = G_a(z_1, z_2) = (z_2, (z_1 + a_1)z_2^2), \quad a_1 \in \mathbb{C}^*, \quad a = (0, a_1)$$



A germ of isomorphism φ which conjugates G_a and $G_{a'}$ leaves the line $\{z_2 = 0\}$ invariant, therefore φ has the form $\varphi(z) = (\varphi_1(z), Bz_2(1 + \theta(z)))$. A simple computation shows that if G_a and $G_{a'}$ are conjugated then

$$a_1 = \pm a'_1.$$

Besides if we want to determine the twisting parameter κ such that $H^0(S, K^{-1} \otimes L^\kappa) \neq 0$, we have to solve the equation

$$\mu(G_a(z)) = \kappa \det DG_a(z) \mu(z).$$

Using the relation $D_{-K} = D_\theta + D$ of [9] or by a direct computation, we know that a section μ of the twisted anticanonical bundle has to vanish at order two along the cycle, i.e. along $\{z_2 = 0\}$, therefore $\mu(z) = z_2^2 A(z) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$ where $A(0) \neq 0$. A straightforward computation or lemma 4.35 shows that

$$\kappa = -a_1^2.$$

By [9] the relation between the twisting parameters κ and $\tilde{\lambda}$ chosen so that $H^0(S, \Theta \otimes L^{\tilde{\lambda}}) \neq 0$, is $\tilde{\lambda} = k\kappa$. Here $k = k(S) = 2$, therefore $\tilde{\lambda} = -2a_1^2$ and

- There is a non-trivial global vector field if and only if $\tilde{\lambda} = 1$ if and only if

$$a_1^2 = -\frac{1}{2}$$

- Since $\lambda = 1/\tilde{\lambda} \in \mathbb{C}^*$ is a parameter of the coarse moduli space, G_a is conjugated to $G_{a'}$ if and only if the corresponding surfaces $S(G_a)$ and $S(G_{a'})$ are isomorphic if and only if $a_1^2 = a'^2_1$. In particular the mapping $\mathbb{C}^* = B_{J,M} \rightarrow B_{2,1,1} = \mathbb{C}^*$ is 2-sheeted non ramified covering space

We are now looking for the missing parameter: we choose $\sigma(z) = (z_1 + \xi z_2 + a_1, z_2)$; the infinitesimal deformation is

$$X(u_1, v_1) = v_1 \frac{\partial}{\partial u_1}.$$

With

$$G_{a,\xi}(z_1, z_2) = (z_2, (z_1 + \xi z_2 + a_1)z_2^2)$$

the same computation gives $\kappa = -a_1^2$. With a fixed such that $a_1^2 = -1/2$ (in order to have a global vector field), the conjugation relation

$$\varphi(G_{a,\xi}(z)) = F_c(\varphi(z))$$

yields the relations

$$\begin{cases} (I) & \varphi_1(z_2, (z_1 + \xi z_2 + a_1)z_2^2) = Bz_2[(1 + \varphi_1(z))(1 + \theta(z)) + cBz_2(1 + \theta(z))^2] \\ (II) & (z_1 + \xi z_2 + a_1)(1 + \theta(z_2, (z_1 + \xi z_2 + a_1)z_2^2)) = B(1 + \theta(z))^2 \end{cases}$$

we compare the homogeneous parts of the same degree

- till degree two and homogeneous part $z_1 z_2^2$ in (I)
- till degree one and homogeneous part z_1^2 in (II)

A straightforward computation with $a_1^2 = -1/2$ yields

$$c = \xi + 2.$$

Therefore all surfaces with global vector fields are obtained when ξ moves in \mathbb{C} , and X acts by translation. In particular when $b_2(S) = 2$, all surfaces are obtained by simple birational

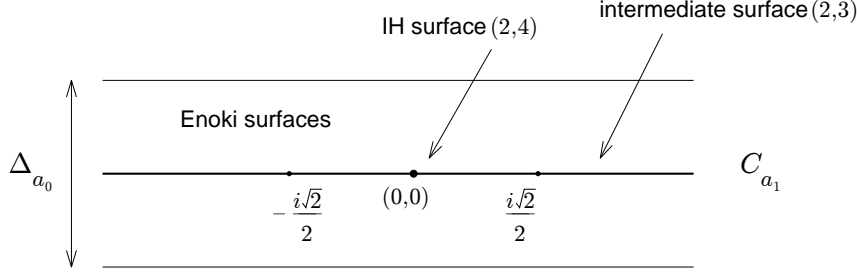
mappings obtained by composition of blowing-ups and an affine map at a suitable place. We extend the family to Enoki surfaces. The family of marked surfaces $\Phi_{J,\sigma} : S_{J,\sigma} \rightarrow B_J$ is defined by the family of polynomial germs

$$G_a(z_1, z_2) = (z_2, z_2^2(z_1 + a_1) + a_0 z_2), \quad a = (a_0, a_1)$$

We have

$$\text{tr}(S) = \text{tr} DG_a(0) = a_0$$

therefore $|a_0| < 1$. The open set $B_J = \Delta_{a_0} \times \mathbb{C}_{a_1}$ has the following strata



Notice that for $\varphi(z_1, z_2) = (-z_1, -z_2)$,

$$\varphi \circ G_{(a_0, a_1)} \circ \varphi^{-1} = G_{(a_0, -a_1)},$$

therefore there is an involution

$$i : B_J \rightarrow B_J, \quad i(a_0, a_1) = (a_0, -a_1),$$

such that G_a and $G_{i(a)}$ give isomorphic surfaces.

Moreover there is also the hypersurface $T_{J,\sigma}$ where there is a relation. The sheaf of relations is generated by global section by theorem A of Cartan. Let

$$\alpha_0(a)[\theta_0] + \beta_0(a)[\mu_0] + \alpha_1(a)[\theta_1] + \beta_1(a)[\mu_1] = 0$$

be such a relation. By the same computation as at the beginning of section 3.3,

$$\beta_0 = \beta_1 = 0,$$

therefore we have to solve the system

$$\begin{cases} X_0 - \Pi_{1*} X_1 &= \alpha_0 \frac{\partial}{\partial u_0} \quad \text{at the point } \Pi_1(u_1, v_1) \\ X_1 - \sigma_* \Pi_{0*} X_0 &= \alpha_1 \frac{\partial}{\partial u'_1} \quad \text{at the point } \sigma \Pi_0(u'_0, v'_0) \end{cases}$$

We have

$$D\Pi_1(u_1, v_1) = \begin{pmatrix} v_1 & u_1 \\ 0 & 1 \end{pmatrix}, \quad D(\sigma\Pi_0)(u'_0, v'_0) = \begin{pmatrix} 0 & 1 \\ v'_0 & u'_0 \end{pmatrix},$$

Since by Hartogs theorem the vector fields X_0 and X_1 extend on the whole blown up ball, they are tangent to the exceptional curves and we set

$$X_0 = A_0 \frac{\partial}{\partial u'_0} + v'_0 B_0 \frac{\partial}{\partial v_0}, \quad X_1 = A_1 \frac{\partial}{\partial u_1} + v_1 B_1 \frac{\partial}{\partial v_1},$$

By a straightforward computation similar to those in the appendix we derive that

$$\alpha_0(a_0, a_1) = 0,$$

for all minimal surfaces, therefore $[\theta_0] \neq 0$ on $\Delta_{a_0} \times \mathbb{C}_{a_1}$ (recall that $\text{tr}(S_a) = a_0$ and the trace is a holomorphic invariant) .

By proposition 3.17, the four cocycles are independent on the line $\{a_0 = 0\}$, hence $\alpha_1(0, a_1) = 0$, and the relation reduces to

$$\alpha_1(a)[\theta_1] = 0$$

with $\alpha_1(0, a_1) = 0$. Therefore $T_{J,\sigma} \cap \{a_0 = 0\} = \{(0, \pm \frac{i\sqrt{2}}{2})\}$, $[\theta_1] = 0$ along $T_{J,\sigma} \setminus \{a_0 = 0\}$ but $[\theta_1] \neq 0$ at the two points where Θ is not locally free. The mapping from the stratum of Enoki surface to the moduli space of Enoki surfaces is discrete and ramified along $T_{J,\sigma}$ and $\{a_1 = 0\}$.

Case $J = \{0, 1\}$

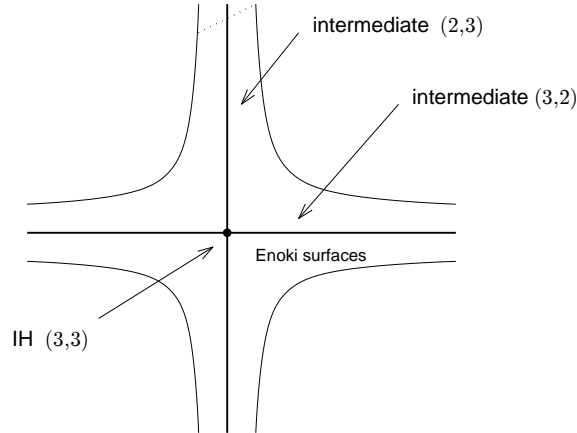
With $\sigma(z) = (z_1 + a_1, z_2)$, the family of marked surfaces $\Phi_{J,\sigma} : S_{J,\sigma} \rightarrow B_J$ is associated to the family of polynomial germs

$$G_a^J(z_1, z_2) = G_a(z_1, z_2) = (z_2(z_1 + a_1), z_2(z_1 + a_1)(z_2 + a_0))$$

$$\det DG_a(z_1, z_2) = z_2^2(z_1 + a_1),$$

$$\text{tr}(S_a) = \text{tr} DG_a(0) = a_0 a_1, \quad \text{with} \quad |a_0 a_1| < 1.$$

There are two lines of intermediate surfaces which meet at the Inoue-Hirzebruch surface with two singular rational curves.



- For $a_0 = 0, a_1 \neq 0, \kappa = a_1$,
- For $a_1 = 0, a_0 \neq 0, \kappa = a_0$.

The involution of the Inoue-Hirzebruch surface which swaps the two cycles induces on the base of the versal family swapping of the two lines of intermediate surfaces.

We have obtained

Theorem 4. 37 *Let $F = \Pi\sigma : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be any holomorphic germ, where $\Pi = \Pi_0\Pi_1$ are blowing-ups and σ is any germ of isomorphism. Then F is conjugated to a birational map obtained by the composition of two blowing-ups*

$$(u, v) \mapsto (uv + a, v), \quad (u', v') \mapsto (v', u'v'),$$

and an affine map at a suitable place. If moreover, S is of intermediate type and there is no non-trivial invariant vector field, F is conjugated to the composition of two blowing-ups of the previous types.

Corollary 4. 38 *Any minimal surface with $b_1(S) = 1$ and $b_2 \leq 2$ containing a GSS admits a birational structure.*

5 Birational germs and new normal forms

5.1 Birational germs of marked surfaces with one tree

Let (S, C_0) be a marked surface with GSS and let M be the intersection matrix of the rational curves. We suppose that C_0 is the root of the unique tree (see picture in section 2.1). Then we have

$$\Pi_l \cdots \Pi_{n-1}(u'', v'') = (u''^p v''^q + a_{l-1}, u''^r v''^s)$$

where $(u'', v'') = (u, v)$ or $(u'', v'') = (u', v')$, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is the composition of matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $A' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, the last one being equal to A' . We set

$$\delta := ps - qr = \pm 1,$$

$$1 \leq d := (r + s) - (p + q) < r + s.$$

Moreover

$$\Pi_0 \cdots \Pi_{l-1}(u, v) = \left(uv^l + \sum_{i=0}^{l-2} a_i v^{i+1}, v \right)$$

Hence

$$F(z) = \Pi\sigma(z) = \left(\sigma_1(z)^{p+rl} \sigma_2(z)^{q+sl} + \sum_{i=0}^{l-1} a_i \left(\sigma_1(z)^r \sigma_2(z)^s \right)^{i+1}, \sigma_1(z)^r \sigma_2(z)^s \right),$$

where σ is a germ of biholomorphism.

If there is no global vector fields the number of parameters given by the blown up points is $2n$ and the expected number of parameters of the versal deformation, therefore the question arises to know if with $\sigma = Id$ we obtain locally versal families. If there are non trivial global vector fields we need (at least) an extra parameter. We add this parameter by the composition $\Pi_0 \cdots \Pi_{l-1} \bar{\sigma} \Pi_l \cdots \Pi_{n-1} Id$ where

$$\bar{\sigma}(u, v) = (u + a_{l+K} v^{l+K}, v), \quad K \geq 0,$$

where K will be chosen in proposition 40. We obtain a new mapping (denoted in the same way)

$$\begin{aligned} F(z) &= \Pi_0 \cdots \Pi_{l-1} \bar{\sigma} \Pi_l \cdots \Pi_{n-1} Id(z) \\ &= \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right). \end{aligned}$$

Now we choose σ such that the following diagram

$$\begin{array}{ccc} W_{n-1} & \xrightarrow{\sigma} & W_{n-1} \\ \Pi_{n-1} \downarrow & & \downarrow \Pi_{n-1} \\ \vdots & & \vdots \\ \Pi_l \downarrow & & \downarrow \Pi_l \\ W_{l-1} & \xrightarrow{\bar{\sigma}} & W_{l-1} \end{array}$$

is commutative, therefore we obtain a one parameter family of birational functions $\sigma = \sigma_{a_{l+K}}$ depending on a_{l+K} such that $F = \Pi\sigma$ is birational and in usual form. We obtain large families $\mathcal{S}_{J, \sigma_{a_{l+K}}} \rightarrow B_J$ and we shall prove that the stratum $B_{J,M}$ is a ramified covering over the OT moduli space of marked surfaces with GSS and intersection matrix M .

Lemma 5. 39 *Let*

$$F(z) = \Pi\sigma(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

then the associated surface $S = S(F)$ admits a non trivial global twisted vector field if and only if

$$u = \frac{p + s + rl - 1 - \delta}{r + s - 1}, \quad v = \frac{r + q + sl - 1 + \delta}{r + s - 1}, \quad \text{where } \delta := ps - qr$$

are positive integers. Moreover this twisted vector field is a global vector field if and only if

$$\delta a_0^u k(S) = 1.$$

Proof: We have by a straightforwad computation

$$\det DF(z) = (ps - qr) z_1^{p+r(l+1)-1} z_2^{q+s(l+1)-1}.$$

By [9], there exists a non trivial global twisted vector field $\theta \in H^0(S, \Theta \otimes L^\lambda)$ on S if and only if there is a global twisted section of the anticanonical bundle $\omega \in H^0(S, K^{-1} \otimes L^\kappa)$. Moreover the twisting factors satisfy the relation $\lambda = k(S)\kappa$. The section θ is a global vector field if $\lambda = 1$ i.e.

$$\kappa = \frac{1}{k(S)} \tag{1}$$

Such a section exists if and only if there is a germ of 2-vector field (denoted in the same way)

$$\omega(z) = z_1^u z_2^v A(z) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$$

where $A(0) \neq 0$ such that $\omega(F(z)) = \kappa \det DF(z) \omega(z)$, or equivalently,

$$(a_0 z_1^r z_2^s + \dots)^u (z_1^r z_2^s)^v A(F(z)) = \kappa (ps - qr) z_1^{p+r(l+1)-1+u} z_2^{q+s(l+1)-1+v} A(z).$$

Comparing terms of lower degree, we obtain the necessary condition

$$a_0^u (z_1^r z_2^s)^{u+v} = \kappa (ps - qr) z_1^{p+r(l+1)-1+u} z_2^{q+s(l+1)-1+v}$$

therefore u and v satisfy the linear system

$$\begin{cases} r(u+v) &= p + r(l+1) - 1 + u \\ s(u+v) &= q + s(l+1) - 1 + v \end{cases}$$

The determinant of the system is $\Delta = -r - s + 1 < 0$ and the solution is

$$u = \frac{p + s + rl - 1 - \delta}{r + s - 1}, \quad v = \frac{r + q + sl - 1 + \delta}{r + s - 1}, \quad \text{where } \delta := ps - qr = \pm 1.$$

Since u and v are the vanishing orders of ω along the curves, a necessary condition for the existence of ω is that u and v are positive integers. Cancelling the common factors we obtain

$$a_0^u = \kappa \delta$$

and with relation (1)

$$\kappa = \delta a_0^u = \frac{1}{k(S)}.$$

If u and v are integers,

$$(a_0 + \dots)^u A(F(z)) = \kappa \delta A(z),$$

with $a_0 \neq 0$. Setting

$$1 + f(z) = \frac{\kappa \delta}{(a_0 + \dots)^u},$$

we have

$$A(F(z)) = (1 + f(z))A(z)$$

Therefore

$$A(z) = \frac{A(0)}{\prod_{j=0}^{\infty} (1 + f(F^j(z)))},$$

the infinite product converges because F is contractant. This proves the existence of ω . \square

Proposition 5. 40 *Let*

$$F(z) = \Pi\sigma(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

and let $S = S(F)$ be the associated surface. Then the surface $S(F)$ admits a non trivial global twisted vector field if and only if there exists an integer $k \geq 0$ such that

$$l = d + k(r + s - 1),$$

If this condition is fulfilled, we choose

$$K = k$$

and $S(F)$ admits a non trivial vector field if and only if for $u = \frac{p+s+rl-1-\delta}{r+s-1} \in \mathbb{N}^*$,

$$\delta a_0^u k(S) = 1.$$

Proof: With notations of lemma 5.39, we have to show that u and v are integers if and only if $l = d + k(r + s - 1)$.

If u and v are integers,

$$u + v = l + 1 + \frac{p + q + l - 1}{r + s - 1} \in \mathbb{N},$$

where $p + q < r + s$. Therefore, $l = d + k(r + s - 1)$. Conversely, if $l = d + k(r + s - 1)$, it is easy to check that u and v are integers and the proof is left to the reader. \square

Proposition 5. 41 *Let*

$$F(z) = \Pi\sigma(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

and $S = S(F)$ the associated surface. Then

$$k(S) = r + s.$$

Proof: The dual graph of the curves is composed of a cycle with (here) only one chain of rational curves called the tree. The proof is achieved by induction on the number $N \geq 1$ of singular sequences. We denote as in [5]

$$a(S) = (s_{k_1} \cdots s_{k_N} r_l),$$

where for any $k \geq 1$, s_k is the singular k -sequence $s_k = (k+2, 2, \dots, 2)$ and r_l is the regular l -sequence $r_l = (2, \dots, 2)$. We have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_N \end{pmatrix}$$

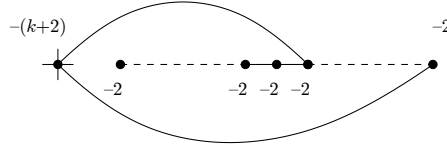
and for any $1 \leq i \leq N$ we set

$$\begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix} = \begin{pmatrix} p_i(k_1, \dots, k_i) & q_i(k_1, \dots, k_i) \\ r_i(k_1, \dots, k_i) & s_i(k_1, \dots, k_i) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_i \end{pmatrix},$$

therefore

$$\begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix} = \begin{pmatrix} q_{i-1} & p_{i-1} + k_i q_{i-1} \\ s_{i-1} & r_{i-1} + k_i s_{i-1} \end{pmatrix} \quad (2)$$

If $N = 1$, dual graph of the curves is



the (opposite) intersection matrix of the (unique) tree is the matrix of a chain of length k

$$\delta_k = \begin{vmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{vmatrix}$$

We have $\delta_k = k+1$ and by [7], $k(S)$ is equal to δ_k . Now here

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}$$

therefore the result is checked for $N = 1$.

If $N = 2$, the sequence of opposite self-intersections of the curves in the tree is

$$\underbrace{2 \cdots 2}_{k_1-1} (k_2 + 2)$$

On one hand

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & k_2 \end{pmatrix} = \begin{pmatrix} 1 & k_2 \\ k_1 & 1 + k_1 k_2 \end{pmatrix}$$

On second hand, the order of the (opposite) intersection matrix of the tree is k_1 . By [7],

$$k(S) = \begin{vmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & k_2 + 2 & \end{vmatrix} = k_1 k_2 + k_1 + 1 = r + s.$$

- If $N = 2\nu$, the sequence of opposite self-intersections of the curves in the tree is

$$\underbrace{2 \cdots 2}_{k_1-1} (k_2 + 2) \underbrace{2 \cdots 2}_{k_3-1} \cdots \cdots \cdots \underbrace{2 \cdots 2}_{k_{2\nu-1}-1} (k_{2\nu} + 2)$$

- If $N = 2\nu + 1$, the sequence of opposite self-intersections is

$$\underbrace{2 \cdots 2}_{k_1-1} (k_2 + 2) \underbrace{2 \cdots 2}_{k_3-1} \cdots \cdots \cdots \underbrace{2 \cdots 2}_{k_{2\nu-1}-1} (k_{2\nu} + 2) \underbrace{2 \cdots 2}_{k_{2\nu+1}}$$

- If $N = 2\nu$, we have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_{2\nu} \end{pmatrix}$$

the determinant of the opposite self-intersection matrix of the tree is

$$\delta(k_1, \dots, k_{2\nu}) = \begin{vmatrix} \boxed{D} & & -1 \\ & -1 & \\ & & k_{2\nu} + 2 \end{vmatrix}$$

where $D = D(k_1, \dots, k_{2\nu-1})$ is the block corresponding to

$$\underbrace{2 \cdots 2}_{k_1-1} (k_2 + 2) \underbrace{2 \cdots 2}_{k_3-1} \cdots \cdots \cdots \underbrace{2 \cdots 2}_{k_{2\nu-1}-1}$$

We have by [7], the induction hypothesis and relations (2),

$$\begin{aligned} k(S) &= \delta(k_1, \dots, k_{2\nu}) = k_{2\nu} \det D(k_1, \dots, k_{2\nu-1}) + \det D(k_1, \dots, k_{2\nu-1} + 1) \\ &= k_{2\nu} \left(r(k_1, \dots, k_{2\nu-2}, k_{2\nu-1} - 1) + s(k_1, \dots, k_{2\nu-2}, k_{2\nu-1} - 1) \right) \\ &\quad + r(k_1, \dots, k_{2\nu-1}) + s(k_1, \dots, k_{2\nu-1}) \\ &= k_{2\nu} \left(r(k_1, \dots, k_{2\nu-2}, k_{2\nu-1}) + s(k_1, \dots, k_{2\nu-2}, k_{2\nu-1}) - s(k_1, \dots, k_{2\nu-2}) \right) \\ &\quad + r(k_1, \dots, k_{2\nu-1}) + s(k_1, \dots, k_{2\nu-1}) \\ &= k_{2\nu} s(k_1, \dots, k_{2\nu-2}, k_{2\nu-1}) + r(k_1, \dots, k_{2\nu-1}) + s(k_1, \dots, k_{2\nu-1}) \\ &= r(k_1, \dots, k_{2\nu}) + s(k_1, \dots, k_{2\nu}) = r + s. \end{aligned}$$

- If $N = 2\nu + 1$, we follow similar arguments:

Let D be the matrix of the chain

$$\underbrace{2 \cdots 2}_{k_1-1} (k_2 + 2) \underbrace{2 \cdots 2}_{k_3-1} \cdots \cdots \cdots \underbrace{2 \cdots 2}_{k_{2\nu-3}-1} (k_{2\nu-2} + 2)$$

then by [7], $k(S) = \delta(k_1, \dots, k_{2\nu+1})$ and

$$\begin{aligned}
\delta(k_1, \dots, k_{2\nu+1}) &= \left| \begin{array}{c|c} \boxed{D} & \begin{array}{c} -1 \\ \hline \begin{array}{cccccc} 2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & 2 & -1 & & \\ & & \ddots & k_{2\nu} + 2 & \ddots & \\ & & & -1 & 2 & \ddots \\ & & & & \ddots & \ddots & -1 \\ & & & & & -1 & 2 \end{array} \end{array} \end{array} \right| \begin{array}{l} 1 \\ \sum_{i=1}^{\nu-1} k_{2i-1} \\ \\ \sum_{i=1}^{\nu} k_{2i-1} \\ \\ \sum_{i=1}^{\nu+1} k_{2i-1} \end{array} \\
&= k_{2\nu}(k_{2\nu+1} + 1) \left| \begin{array}{c|c} \boxed{D} & \begin{array}{c} -1 \\ \hline \begin{array}{cccc} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{array} \end{array} \end{array} \right| \begin{array}{l} 1 \\ \sum_{i=1}^{\nu-1} k_{2i-1} \\ \sum_{i=1}^{\nu-1} k_{2i-1} + 1 \\ \sum_{i=1}^{\nu} k_{2i-1} - 1 \end{array} \\
&+ \left| \begin{array}{c|c} \boxed{D} & \begin{array}{c} -1 \\ \hline \begin{array}{cccc} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{array} \end{array} \end{array} \right| \begin{array}{l} 1 \\ \sum_{i=1}^{\nu-1} k_{2i-1} \\ \sum_{i=1}^{\nu-1} k_{2i-1} + 1 \\ \sum_{i=1}^{\nu} k_{2i+1} \end{array} \\
&= k_{2\nu}(k_{2\nu+1} + 1) \delta(k_1, \dots, k_{2\nu-2}, k_{2\nu-1} - 1) + \delta(k_1, \dots, k_{2\nu-2}, k_{2\nu-1} + k_{2\nu+1}) \\
&= k_{2\nu}(k_{2\nu+1} + 1) (r_{2\nu-1} + s_{2\nu-1} - s_{2\nu-2}) + s_{2\nu-2} + r_{2\nu-2} \\
&\quad + (k_{2\nu-1} + k_{2\nu+1}) s_{2\nu-2}.
\end{aligned}$$

A straightforward computation show that this last expression is equal to $r_{2\nu+1} + s_{2\nu+1}$. □

Corollary 5. 42 *The index of the surface $S(F)$ is*

$$Index(S) = \frac{r + s - 1}{gcd\{r + s - 1, p + q + l - 1\}}.$$

Corollary 5. 43 *Suppose that $l = d + k(r + s - 1)$, then S admits a non trivial global vector field if and only if*

$$1 - \delta(r + s) a_0^{(k+1)r-p+1} = 0.$$

Proof: If $l = d + k(r + s - 1)$, it is easy to check that

$$u = \frac{p + s + rl - 1 - \delta}{r + s - 1} = (k + 1)r - p + 1.$$

By propositions 5.40 and 5.41, we have the result. \square

Notations 5. 44 We denote by $\mathcal{G} = \mathcal{G}(p, q, r, s, l)$ the family of contracting birational mappings

$$G(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

where $a_0 \in \mathbb{C}^*$, $a_i \in \mathbb{C}$, $i = 1, \dots, l-1, l+K$, $a_{l+K} = 0$ if there is no integer k such that $l = d + k(r + s - 1)$ and by $\Phi = \Phi(p, q, r, s, l)$ the set of the germs of biholomorphisms $\varphi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ for which there exists $G, G' \in \mathcal{G}$ such that $G' = \varphi^{-1} G \varphi \in \mathcal{G}$. Let $L := L(p, q, r, s, l)$ be the group of diagonal linear mappings $\varphi_{A,B}(z_1, z_2) = (Az_1, Bz_2)$ where A, B satisfy the condition

$$B = A^r B^s, \quad A = A^{p+rl} B^{q+sl}$$

Lemma 5. 45 1) The group L is a subgroup of $\mathbb{U}_{p+s+rl-\delta-1} \times \mathbb{U}_{p+s+rl-\delta-1}$, where for any $m \in \mathbb{N}^*$, \mathbb{U}_m is the group of m -roots of unity.

2) The group L operates on \mathcal{G} ; more precisely if $\varphi_{A,B} \in L$ and

$$G(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

then

$$G'(z) = \varphi_{A,B}^{-1} G \varphi_{A,B}(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a'_i (z_1^r z_2^s)^{i+1} + a'_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

where

$$A a'_i = B^{i+1} a_i, \quad \text{for } i = 0, \dots, l-1, l+K.$$

In particular L is an abelian group contained in Φ .

The proof is easy and left to the reader. \square

5.2 Moduli spaces of birational mappings

We want to determine the equivalence classes of the birational mappings G , previously defined or, that is equivalent, the fibers of the canonical morphism to the OT moduli space. Let

$$G(z) = \Pi \sigma(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

$$G'(z) = \Pi' \sigma'(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a'_i (z_1^r z_2^s)^{i+1} + a'_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right)$$

be two such birational germs and suppose that there exists a germ of biholomorphism φ such that $G' \circ \varphi = \varphi \circ G$. Since the degeneration set $\{z_1 z_2 = 0\}$ is invariant and φ cannot swap the rational curves, φ has the form

$$\varphi(z_1, z_2) = (Az_1(1 + \theta(z)), Bz_2(1 + \mu(z))).$$

We have

$$\begin{aligned} \varphi(G(z)) &= \left(A \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i \in \{0, \dots, l-1, l+K\}} a_i (z_1^r z_2^s)^{i+1} \right) (1 + \theta(G(z))), B z_1^r z_2^s (1 + \mu(G(z))) \right) \\ G'(\varphi(z)) &= \left(\left(A z_1 (1 + \theta(z)) \right)^{p+rl} \left(B z_2 (1 + \mu(z)) \right)^{q+sl} \right. \\ &\quad + \sum_{i \in \{0, \dots, l-1, l+K\}} a'_i \left(A z_1 (1 + \theta(z)) \right)^{r(i+1)} \left(B z_2 (1 + \mu(z)) \right)^{s(i+1)}, \\ &\quad \left. A^r B^s z_1^r z_2^s (1 + \theta(z))^r (1 + \mu(z))^s \right) \end{aligned}$$

Second members yield the equality

$$(II) \quad B(1 + \mu(G(z))) = A^r B^s (1 + \theta(z))^r (1 + \mu(z))^s$$

Therefore

$$B = A^r B^s, \quad \text{and} \quad 1 + \mu(z) = \left(\prod_{j=0}^{\infty} (1 + \theta(G^j(z)))^{r/s^{j+1}} \right)^{-1}. \quad (3)$$

First members of the conjugation give

$$(I) \quad \left\{ \begin{aligned} & A \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i \in \{0, \dots, l-1, l+K\}} a_i (z_1^r z_2^s)^{i+1} \right) (1 + \theta(G(z))) \\ &= \left(A z_1 (1 + \theta(z)) \right)^{p+rl} \left(B z_2 (1 + \mu(z)) \right)^{q+sl} \\ &\quad + \sum_{i \in \{0, \dots, l-1, l+K\}} a'_i \left(A z_1 (1 + \theta(z)) \right)^{r(i+1)} \left(B z_2 (1 + \mu(z)) \right)^{s(i+1)} \end{aligned} \right.$$

Setting $\delta = ps - qr = \pm 1$, we obtain with (3),

$$(I) \quad \left\{ \begin{aligned} & A \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i \in \{0, \dots, l-1, l+K\}} a_i (z_1^r z_2^s)^{i+1} \right) (1 + \theta(G(z))) \\ &= A^{p+rl} B^{q+sl} z_1^{p+rl} z_2^{q+sl} (1 + \theta(z))^{\delta/s} \left(\prod_{j=1}^{\infty} (1 + \theta(G^j(z)))^{r/s^{j+1}} \right)^{-(q+sl)} \\ &\quad + \sum_{i \in \{0, \dots, l-1, l+K\}} a'_i B^{i+1} z_1^{r(i+1)} z_2^{s(i+1)} \left(\prod_{j=1}^{\infty} (1 + \theta(G^j(z)))^{r/s^{j+1}} \right)^{-s(i+1)} \end{aligned} \right.$$

The following lemma is evident:

Lemma and Definition 5. 46 *The positive integral solutions of the system*

$$(E) \quad \begin{cases} p + rl + \alpha &= rj \\ q + sl + \beta &= sj \end{cases}$$

are all of the form

$$\begin{cases} \alpha &= kr - p \\ \beta &= ks - q \end{cases}, \quad k \geq 1.$$

In particular the least solution is $(r-p, s-q)$. When (E) has a solution we shall say that there is a **resonance**.

Comparing monomial terms $z_1^{p+rl} z_2^{q+sl}$ in (I) we obtain thanks to lemma 5.46

$$A = A^{p+rl} B^{q+sl} \quad (4)$$

By lemma 5.45, A, B are roots of unity.

Let $Aut(\mathbb{C}^2, 0)$ be the group of germs of biholomorphisms of $(\mathbb{C}^2, 0)$ and $Aut(\mathbb{C}^2, H, 0)$ be the subgroup of $Aut(\mathbb{C}^2, 0)$ which leave each of the components of the hypersurface $H = \{z_1 z_2 = 0\}$ invariant, i.e. $\varphi \in G$ has the form

$$\varphi(z) = (Az_1(1 + \theta(z)), Bz_2(1 + \mu(z))).$$

Let $Aut(\mathbb{C}^2, H, 0)_I$ be the subgroup of $Aut(\mathbb{C}^2, 0)$ of germs of biholomorphisms φ tangent to the identity, i.e.

$$\varphi(z) = (z_1(1 + \theta(z)), z_2(1 + \mu(z))).$$

Lemma 5. 47 *Let $\alpha : Aut(\mathbb{C}^2, H, 0)_I \rightarrow Aut(\mathbb{C}^2, H, 0)$ the canonical injection and $\beta : Aut(\mathbb{C}^2, H, 0) \rightarrow L$ defined by $\beta(\varphi) = \varphi_{AB}$. Then, $Aut(\mathbb{C}^2, H, 0)_I$ is a normal subgroup of $Aut(\mathbb{C}^2, H, 0)$ and we have the exact sequence*

$$\{Id\} \rightarrow Aut(\mathbb{C}^2, H, 0)_I \xrightarrow{\alpha} Aut(\mathbb{C}^2, H, 0) \xrightarrow{\beta} L \rightarrow \{Id\}.$$

Replacing φ by $\varphi\varphi_{A,B}^{-1}$ we obtain an automorphism tangent to the identity, therefore we have to determine equivalence classes of the equivalence relation on \mathcal{G}

$$G \sim G' \iff \exists \varphi \in Aut(\mathbb{C}^2, H, 0)_I, \quad G'\varphi = \varphi G.$$

The equation (I) becomes

$$(I) \quad \left\{ \begin{aligned} & \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i \in \{0, \dots, l-1, l+K\}} a_i (z_1^r z_2^s)^{i+1} \right) (1 + \theta(G(z))) \\ &= z_1^{p+rl} z_2^{q+sl} (1 + \theta(z))^{\delta/s} \left(\prod_{j=1}^{\infty} (1 + \theta(G^j(z)))^{r/s^{j+1}} \right)^{-(q+sl)} \\ &+ \sum_{i \in \{0, \dots, l-1, l+K\}} a'_i (z_1^r z_2^s)^{i+1} \left(\prod_{j=1}^{\infty} (1 + \theta(G^j(z)))^{r/s^{j+1}} \right)^{-s(i+1)} \end{aligned} \right.$$

The question is to determine the quotient \mathcal{G}/\sim . We shall see at the end of this section that the equivalence relation is generically trivial.

Lemma 5. 48 *Let $\mu = \max \left\{ d, l + \left\lceil \frac{l-d}{r+s-1} \right\rceil \right\}$ and $\theta(z) = \sum_{i+j \geq 1} t_{ij} z_1^i z_2^j$. If $t_{ij} = 0$ for $i+j \leq \mu$, then*

$$\theta = 0$$

and φ is linear.

Proof: By hypothesis we have

$$\theta(G(z)) = \left(\sum_{i+j=\mu+1} a_0^i t_{ij} \right) (z_1^r z_2^s)^{\mu+1} \mod \mathfrak{M}^{(r+s)(\mu+1)+1}.$$

We show by induction on $k = i + j \geq \mu + 1$ that $t_{ij} = 0$.

We consider the terms of degree $p + q + (r + s)l + \mu + 1$

$$z_1^{p+rl} z_2^{q+sl} \frac{\delta}{s} \sum_{i+j=\mu+1} t_{ij} z_1^i z_2^j$$

and we are looking for other terms of the same degree or bidegrees in (I).

The inequalities

$$\begin{cases} (r + s)l = p + q + (r + s - 1)l + d < p + q + (r + s)l + \mu + 1, \\ p + q + (r + s)l + \mu + 1 < p + q + (r + s)l + (r + s)(\mu + 1), \\ p + q + (r + s)l + \mu + 1 < r + s + (r + s)(\mu + 1) \end{cases}$$

show that there is no other term of the same degree when $a_{K+l} = 0$. If $a_{K+l} \neq 0$, $l = d + K(r + s - 1)$ and it is easy to check that $(r + s)(l + K) \neq p + q + (r + s)l + \mu + 1$, hence $t_{ij} = 0$ if $i + j = \mu + 1$.

Suppose that for $k \geq \mu + 2$,

$$\theta(z) = \sum_{i+j \geq k} t_{ij} z_1^i z_2^j,$$

then the similar inequalities show the result. \square

Lemma 5. 49 *Let $\mu = \max \left\{ d, l + \left\lceil \frac{l - d}{r + s - 1} \right\rceil \right\}$.*

Then, the coefficients t_{ij} , for $i + j \leq \mu$, with a_i and a'_i , $i = 0, \dots, l - 1, l + K$, determine uniquely θ hence also φ .

Proof: We show by induction on $k \geq 0$ that the coefficients t_{ij} for $i + j \leq \mu$ determine uniquely the coefficients t_{ij} for $i + j \geq \mu + k$. It is sufficient to show that if the coefficients t_{ij} , for $i + j \leq \mu + k$ are determined by coefficients t_{ij} for $i + j \leq \mu$ then the coefficients t_{ij} for $i + j = \mu + k + 1$ are determined by coefficients t_{ij} for $i + j \leq \mu + k$. On that purpose we consider homogeneous part of degree $p + q + (r + s)l + \mu + k + 1$ in (I) which contain the part

$$z_1^{p+rl} z_2^{q+sl} \frac{\delta}{s} \left(\sum_{i+j=\mu+k+1} t_{ij} z_1^i z_2^j \right)$$

In order to prove that all other terms with such degree involve only t_{ij} with $i + j \leq \mu + k$, it is sufficient to prove that if $i + j \geq \mu + k + 1$ then

$$r + s + (i + j)(r + s) > p + q + (r + s)l + \mu + k + 1,$$

and it is sufficient to prove that

$$r + s + (\mu + k + 1)(r + s) > p + q + (r + s)l + \mu + k + 1.$$

- If $l \leq d$, $\mu = d$, and we have to check that

$$r + s + (d + k + 1)(r + s) > p + q + (r + s)d + d + k + 1$$

which is clear;

- If $d + K(r + s - 1) \leq l < d + (K + 1)(r + s - 1)$, then $\mu = l + K$. We have to check

$$r + s + (l + K + k + 1)(r + s) > p + q + (r + s)l + l + K + k + 1$$

However this inequality is equivalent to

$$d + (K + k + 1)(r + s - 1) > l$$

which is satisfied by assumption. □

Proposition 5. 50 *Let $\mathcal{G} = \mathcal{G}(p, q, r, s, l)$ the family of contracting birational mappings*

$$G(z) = \left(z_1^{p+rl} z_2^{q+sl} + \sum_{i=0}^{l-1} a_i (z_1^r z_2^s)^{i+1} + a_{l+K} (z_1^r z_2^s)^{l+K+1}, z_1^r z_2^s \right),$$

where $a_0 \in \mathbb{C}^*$, $a_i \in \mathbb{C}$, $i = 1, \dots, l-1, l+K$, $a_{l+K} = 0$ if there is no non trivial twisted vector fields.

1) *The surfaces $S(G)$ have no twisted vector fields, i.e. $l - d \not\equiv 0 \pmod{r + s - 1}$, if and only if $\text{Aut}(\mathbb{C}^2, H, 0)_I \cap \Phi = \{Id\}$.*

2) *If $l = d + K(r + s - 1)$, then $\text{Aut}(\mathbb{C}^2, H, 0)_I \cap \Phi$ is a group isomorphic to $(\mathbb{C}, +)$ and*

a) *If there are global vector fields, $\text{Aut}(\mathbb{C}^2, H, 0)_I \cap \Phi$ acts trivially on \mathcal{G} , in particular a_{l+K} is an effective parameter,*

b) *If there are no global vector fields, $\text{Aut}(\mathbb{C}^2, H, 0)_I \cap \Phi$ acts transitively on $\mathbb{C}_{a_{l+K}}$, i.e. the complex structure on $S(G)$ does not depend on a_{l+K} .*

Proof: Suppose that $\theta \neq 0$ and let $\gamma = \min\{i + j \geq 1 \mid t_{ij} \neq 0\}$. By lemma 48, $\gamma \leq \mu$. The homogeneous parts of lower degree in (I) which involve t_{ij} with $\gamma = i + j$ are

- Case $\gamma \leq l - 1$ or $\gamma = l + K$,

$$(A) \quad a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma + a_\gamma (z_1^r z_2^s)^{\gamma+1},$$

$$(B) \quad z_1^{p+rl} z_2^{q+sl} \frac{\delta}{s} \sum_{i+j=\gamma} t_{ij} z_1^i z_2^j,$$

$$(C) \quad -\frac{r}{s} a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma + a'_\gamma (z_1^r z_2^s)^{\gamma+1}$$

- Case $\gamma \geq l$ and $\gamma \neq l + K$, (A) is replaced by

$$(A') \quad a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma,$$

and (C) by

$$(C') \quad -\frac{r}{s} a_0 z_1^r z_2^s \left(\sum_{i+j=\gamma} t_{ij} a_0^i \right) (z_1^r z_2^s)^\gamma$$

- If there is no resonance, the bidegrees of the terms (A) and (C) (resp. (A') and (C')) are all distinct of those in (B) , therefore we obtain readily

$$\sum_{i+j=\gamma} t_{ij} z_1^i z_2^j = 0,$$

hence a contradiction

- Therefore there is a resonance and there exists a unique coefficient $t_{kr-p, ks-q} \neq 0$ with $k(r+s) - (p+q) = \gamma$. Then
 - Case $\gamma \leq l-1$ or $\gamma = l+K$,

$$\begin{aligned} & a_0 z_1^T z_2^S t_{kr-p, ks-q} a_0^{kr-p} (z_1^T z_2^S)^\gamma + a_\gamma (z_1^T z_2^S)^{\gamma+1} \\ &= z_1^{p+rl} z_2^{q+sl} \frac{\delta}{s} t_{kr-p, ks-q} z_1^{kr-p} z_2^{ks-q} + a'_\gamma (z_1^T z_2^S)^{\gamma+1} - \frac{r}{s} a_0 z_1^T z_2^S t_{kr-p, ks-q} a_0^{kr-p} (z_1^T z_2^S)^\gamma \end{aligned}$$

- Case $\gamma \geq l$, $\gamma \neq l+K$

$$\begin{aligned} & a_0 z_1^T z_2^S t_{kr-p, ks-q} a_0^{kr-p} (z_1^T z_2^S)^\gamma \\ &= z_1^{p+rl} z_2^{q+sl} \frac{\delta}{s} t_{kr-p, ks-q} z_1^{kr-p} z_2^{ks-q} - \frac{r}{s} a_0 z_1^T z_2^S t_{kr-p, ks-q} a_0^{kr-p} (z_1^T z_2^S)^\gamma \end{aligned}$$

The equality of degrees implies that $l = d + (k-1)(r+s-1)$, i.e. the surface has twisted vector fields and $\gamma = l + (k-1) = l+K$ (and the second case never appears). After simplification, we obtain

$$a'_{l+K} = a_{l+K} - t_{kr-p, ks-q} \frac{\delta}{s} \left(1 - \delta(r+s) a_0^{kr-p+1} \right)$$

Let $\mathfrak{M} = (z_1, z_2)$. Since $\theta(z) = 0 \bmod \mathfrak{M}^{l+K}$, (I) gives

$$a'_i = a_i, \quad \text{for } i = 0, \dots, l-1,$$

therefore, applying corollary 43,

- If there are global vector fields, $1 - \delta(r+s) a_0^{kr-p+1} = 0$ and $a'_{l+K} = a_{l+K}$, hence $\text{Aut}(\mathbb{C}^2, H, 0)_I \cap \Phi$ acts trivially;
- If there are no vector fields, $1 - \delta(r+s) a_0^{kr-p+1} \neq 0$, and $G_I \cap \Phi$ acts transitively on the line $\mathbb{C}_{a_{l+K}}$.

By lemma 49, $t = t_{(K+1)r-p, (K+1)s-q} \in \mathbb{C}$ determines the formal series θ . It remains to prove that θ is convergent hence $\text{Aut}(\mathbb{C}^2, H, 0)_I \cap \Phi \simeq \mathbb{C}$.

- If there are global vector fields, there exists a 1-parameter group of automorphisms, therefore there are such θ and conversely, any θ defines an automorphism of $S(G)$ which is in the identity component of $\text{Aut}(S(G))$;
- If there is no global vector fields, a_{l+K} is a superfluous parameter and all surfaces are isomorphic, therefore there are such isomorphisms.

□

5.3 Representation of any marked surface by a birational germ

We want to compare birational germs and Favre polynomial germs of the form

$$F(z_1, z_2) = (\lambda z_1 z_2^\sigma + P(z_2) + c z_2^{\frac{\sigma k}{k-1}}, z_2^k), \quad P(z_2) = \sum_{i=p+q}^{\sigma} b_i z_2^i$$

given in [25] (see section 4.1). The condition $j < k$ (or $p + q < r + s$ in our notations) implies that the first blowing-up is of the form $(u', v') \mapsto (v', u'v')$ hence we have to consider the germ $\Pi_l \cdots \Pi_{n-1} \Pi_0 \cdots \Pi_{l-1} \bar{\sigma}$ at the point $(a_{l-1}, 0)$. After a change of coordinates $u = z_1 + a_{l-1}$, $v = z_2$ we obtain

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^r z_2^s \right)$$

We shall suppose that $a_{l+K} = 0$ if there is no global vector fields i.e. $l - d \not\equiv 0 \pmod{k-1}$ or $\lambda \neq 1$.

By proposition 4.36, λ is determined by a_0 , more precisely

Proposition 5. 51 *Let (S, C_0) be a marked surface such that the dual graph of the curves contains only one tree and C_0 is the (unique) root in the cycle of rational curves. Let G and F be respectively the birational germ and the Favre germ associated to (S, C_0) . Then,*

1) *If μ is the index of S and $\delta = ps - qr$*

$$\kappa = \delta^\mu a_0^{\mu \left[\frac{r\sigma}{k-1} - (p-1) \right]}.$$

2) *If λ is the parameter in Favre normal forms, λ is determined up to a root of unity, more precisely*

$$\lambda^{-1} = \epsilon^\sigma \delta k a_0^{1-p+\frac{r\sigma}{k-1}}, \quad \epsilon^{k-1} = 1.$$

Proof: By lemma 4.35 and proposition 5.41,

$$\kappa = \delta^\mu a_0^{\mu \left[\frac{r\sigma}{k-1} - (p-1) \right]}.$$

Applying 1), lemmas 4.31 and 4.23,

$$\lambda^{-1} = \epsilon^\sigma \delta k(S) a_0^{1-p+\frac{r\sigma}{k-1}}.$$

□

The aim of the sequel of this section is to prove

Theorem 5. 52 *We choose $a_0 \in \mathbb{C}^*$, ϵ such that $\epsilon^{r+s-1} = 1$, and let $\sigma = p + q + l - 1$. Then*
A) If $r + s - 1$ does not divide $l - d$ or $\lambda \neq 1$ there is a bijective polynomial mapping

$$\begin{aligned} f_{a_0, \epsilon} : \quad \mathbb{C}^{l-1} &\longrightarrow \mathbb{C}^{l-1} \\ a = (a_1, \dots, a_{l-1}) &\longmapsto (b_{p+q+1}(a), \dots, b_{p+q+l-1}(a)) \end{aligned}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} \right)^r z_2^s \right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^\sigma + \sum_{i=p+q}^{\sigma} b_i z_2^i, z_2^{r+s} \right),$$

where λ depends only on a_0 by 5.51.

B) If $l - d = K(r + s - 1)$ and $\lambda = 1$, there is a bijective polynomial mapping

$$\begin{aligned} f_{a_0, \epsilon} : \quad \mathbb{C}^{l-1} \times \mathbb{C} &\longrightarrow \mathbb{C}^{l-1} \times \mathbb{C} \\ a = (a_1, \dots, a_{l-1}, a_{l+K}) &\longmapsto (b_{p+q+1}(a), \dots, b_{p+q+l-1}(a), c(a)) \end{aligned}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^r z_2^s \right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^\sigma + \sum_{k=p+q}^{\sigma} b_k z_2^k + c z_2^{\frac{\sigma k(S)}{k(S)-1}}, z_2^{r+s} \right).$$

Proof: Let $\varphi(z) = (\varphi_1(z), C z_2(1 + \mu(z)))$ be a germ of biholomorphic map which preserves the degeneration set $\{z_2 = 0\}$.

A) **We suppose that $l - d \not\equiv 0 \pmod{r + s - 1}$ or $\lambda \neq 1$.** We have, since $a_{l+K} = 0$,

$$\varphi(G(z)) = \left(\varphi_1(G(z)), C \left\{ z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} \right\}^r z_2^s (1 + \mu(G(z))) \right),$$

$$F(\varphi(z)) = \left(\lambda \varphi_1(z) C^\sigma z_2^\sigma (1 + \mu(z))^\sigma + \sum_{k=p+q}^{\sigma} b_k C^k z_2^k (1 + \mu(z))^k, C^{r+s} z_2^{r+s} (1 + \mu(z))^{r+s} \right).$$

Comparing right members we have

$$(II) \quad \left\{ z_1 z_2^{l-1} + \sum_{i=0}^{l-1} a_i z_2^i \right\}^r (1 + \mu(G(z))) = C^{r+s-1} z_2^r (1 + \mu(z))^{r+s}.$$

Constant parts give the condition

$$a_0^r = C^{r+s-1} \quad (5)$$

therefore C is determined up to a root of unity ϵ such that $\epsilon^{r+s-1} = 1$. In other terms if we choose a local determination of the $(r + s - 1)$ -root $a_0^{1/(r+s-1)}$,

$$C = \epsilon a_0^{r/(r+s-1)}, \quad \epsilon^{r+s-1} = 1. \quad (6)$$

Moreover the equation

$$(II) \quad \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1} \right) \right\}^r (1 + \mu(G(z))) = (1 + \mu(z))^{r+s}.$$

has the solution

$$1 + \mu(z) = \prod_{j=0}^{\infty} \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i (G_2^j(z))^i + G_1^j(z) (G_2^j(z))^{l-1} \right) \right\}^{\frac{r}{(r+s)j+1}}$$

Left members give the equality

$$(I) \quad \left\{ \varphi_1 \left(a_0^p \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1} \right) \right\}^p z_2^{p+q}, a_0^r \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1} \right) \right\}^r z_2^{r+s} \right) \right. \\ \left. = \lambda \varphi_1(z) \left(C z_2 (1 + \mu(z)) \right)^{p+q+l-1} + \sum_{k=p+q}^{p+q+l-1} b_k C^k z_2^k (1 + \mu(z))^k \right.$$

We want to express the coefficients b_k with the a_i 's, however the coefficients A_{ij} of the series

$$\varphi_1(z_1, z_2) = \sum_{i,j} A_{ij} z_1^i z_2^j$$

depend also on a_i 's. For example, considering homogeneous parts of bidegree $(0, p+q)$, we have,

$$(R_0) \quad A_{10} a_0^p = b_{p+q} C^{p+q} = C^{p+q}$$

hence with (6),

$$A_{10} = \epsilon^{p+q} a_0^{\frac{(p+q)r}{r+s-1} - p}. \quad (7)$$

If $p > 0$, $r+s > p+q+1$ and $l \geq 2$, homogeneous part of bidegree $(0, p+q+1)$ gives

$$A_{10} a_0^{p-1} p a_1 = b_{p+q+1} C^{p+q+1} + C^{p+q} \frac{(p+q)r}{r+s} \frac{a_1}{a_0}$$

therefore by (R_0) ,

$$(R_1) \quad b_{p+q+1} = \frac{\delta a_1}{C a_0 (r+s)}.$$

Comparing terms of bidegree $(1, p+q+l-1)$ we obtain

$$A_{10} p a_0^{p-1} = \lambda A_{10} C^{p+q+l-1} + \frac{r(p+q)}{r+s} \frac{C^{p+q}}{a_0}$$

therefore with (7) and (6), and since $k = k(S) = r+s$,

$$\lambda = \frac{\delta}{\epsilon^\sigma k} a_0^{p-1 - \frac{r\sigma}{k-1}}. \quad (8)$$

where $\delta = ps - qr$.

In order to express the coefficients b_{p+q+j} , $j \geq 1$, as polynomials of variables a_1, \dots, a_{l-1} , it is also necessary to express the coefficients A_{ij} involved in the relations as polynomials of the same variables a_1, \dots, a_{l-1} . Therefore we have to determine the set of points $(i, j) \in \mathbb{N} \times \mathbb{N}$ which occur as indices of the A_{ij} 's in the relations.

Let E_0 be the subset of indices (i, j) which occur in homogeneous part of bidegree $(0, k)$ for $p+q \leq k \leq p+q+l-1$ in equation (I). We have

$$E_0 = \{(i, j) \mid p+q \leq i(p+q) + j(r+s) \leq p+q+l-1\}$$

Then we define a translation

$$T(i, j) = (i, j + p + q + l - 1)$$

and we want to determine which coefficients $A_{\alpha\beta}$ are involved on the homogeneous part of bidegree $T(i, j)$. On that purpose we define a sequence $(E_m)_{m \geq -1}$ of increasing subsets of $\mathbb{N} \times \mathbb{N}$, starting with $E_{-1} = \emptyset$,

$$E_m = \left\{ (i, j) \mid i(p+q) + j(r+s) \leq (p+q+l-1) \left(1 + \frac{1}{r+s} + \dots + \frac{1}{(r+s)^m} \right) \right\}, \quad m \geq 0,$$

and

$$E_\infty := \left\{ (i, j) \mid i(p+q) + j(r+s) < (p+q+l-1) \frac{r+s}{r+s-1} \right\}.$$

Lemma 5. 53 Suppose $l - d \not\equiv 0 \pmod{r + s - 1}$. Let $(i, j) \in E_m$, $m \geq 0$.

1) If $i \geq 2$ then for any (α, β) , the homogeneous parts of bidegree $(i, j + p + q + l - 1)$ satisfy

$$\left\langle A_{\alpha\beta} a_0^{p\alpha+r\beta} \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1} \right) \right\}^{p\alpha+r\beta} z_2^{\alpha(p+q)+\beta(r+s)} \right\rangle_{i,j+p+q+l-1} = 0.$$

2) If $i = 1$, and homogeneous part of bidegree $(i, j + p + q + l - 1)$ satisfies

$$\left\langle A_{\alpha\beta} a_0^{p\alpha+r\beta} \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1} \right) \right\}^{p\alpha+r\beta} z_2^{\alpha(p+q)+\beta(r+s)} \right\rangle_{i,j+p+q+l-1} \neq 0$$

then $(\alpha, \beta) \in E_1$.

Moreover

- If $m = 0$, then $(\alpha, \beta) \in E_0$,
- If $\alpha = 1$, then $\beta \leq j/(r + s)$, in particular if $j \neq 0$, then $\beta \neq j$,
- If $\alpha = 0$, then $\beta < j$ or $\{(i, j) = (1, 1) \text{ and } (\alpha, \beta) = (0, 1)\}$.

3) If $i = 0$ and

$$\left\langle A_{0\beta} a_0^{r\beta} \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1} \right) \right\}^{r\beta} z_2^{\beta(r+s)} \right\rangle_{0,j+p+q+l-1} \neq 0,$$

the following two conditions cannot be fulfilled at the same time

- $i = \alpha = 0$, and $j = \beta$,
- $\beta(r + s) = j + p + q + l - 1$,

i.e. if the coefficient $A_{0,j}$ appears two times when considering homogeneous part of bidegree $(0, j + p + q + l - 1)$, one occurrence is multiplied by a non constant polynomial in a_1, \dots, a_j .

4) If $(i, j) \in E_m$ and $i \geq 2$, then $A_{ij} = 0$.

5) If $(0, j) \in E_m \setminus E_{m-1}$, $m \geq 0$ and (α, β) satisfies

$$\alpha(p + q) + \beta(r + s) = j + p + q + l - 1,$$

then

- $(\alpha, \beta) \in E_{m+1} \setminus E_m$
- $\alpha = 0$ or $\alpha = 1$ and (α, β) is unique.

In other words, in homogeneous part of bidegree $(0, j + p + q + l - 1)$, there are, modulo $\mathfrak{M} = (a_1, \dots, a_{l-1})$, at most two coefficients which occur: $A_{0,j}$ and perhaps another $A_{\alpha\beta}$ with $\alpha = 0$ or $\alpha = 1$.

Proof: If

$$\left\langle A_{\alpha\beta} a_0^{p\alpha+r\beta} \left\{ 1 + \frac{1}{a_0} \left(\sum_{i=1}^{l-1} a_i z_2^i + z_1 z_2^{l-1} \right) \right\}^{p\alpha+r\beta} z_2^{\alpha(p+q)+\beta(r+s)} \right\rangle_{i,j+p+q+l-1} \neq 0$$

then, $p\alpha + r\beta \geq i$, and the least degree in z_2 is

$$(l - 1)i + \alpha(p + q) + \beta(r + s).$$

Since $\langle \cdot \rangle_{i,j+p+q+l-1} \neq 0$,

$$(*) \quad (l-1)i + \alpha(p+q) + \beta(r+s) \leq j + p + q + l - 1$$

however by assumption $(i, j) \in E_m \subset E_\infty$,

$$j < \frac{p+q+l-1}{r+s-1} - i \frac{p+q}{r+s}$$

therefore

$$(**) \quad (l-1)i + i \frac{p+q}{r+s} + \alpha(p+q) + \beta(r+s) < (p+q+l-1) \left(1 + \frac{1}{r+s-1} \right)$$

Notice that $(i, j) = (\alpha, \beta)$ if and only if $(i, j) = (1, 0)$.

1) If $i \geq 2$, we have, by $(**)$,

$$2(l-1) + 2 \frac{p+q}{r+s} + \alpha(p+q) + \beta(r+s) < (p+q+l-1) \left(1 + \frac{1}{r+s-1} \right)$$

Since $p+q < r+s$,

$$(l-1) \left(1 - \frac{1}{r+s-1} \right) + (p+q) \left(\frac{2}{r+s} - \frac{1}{r+s-1} + \alpha + \beta - 1 \right) < 0$$

which is impossible.

2) Suppose that $(i, j) \in E_m$, and $i = 1$ then

$$j < \frac{p+q+l-1}{r+s} \left(1 + \frac{1}{r+s-1} \right) - \frac{p+q}{r+s}$$

and by $(*)$

$$(l-1) + \frac{p+q}{r+s} + \alpha(p+q) + \beta(r+s) < (p+q+l-1) \left(1 + \frac{1}{r+s} + \frac{1}{(r+s)(r+s-1)} \right)$$

which is equivalent to

$$\begin{aligned} (l-1) \left(1 - \frac{1}{(r+s)(r+s-1)} \right) + (p+q) \left(\frac{1}{r+s} - \frac{1}{(r+s)(r+s-1)} \right) + \alpha(p+q) + \beta(r+s) \\ < (p+q+l-1) \left(1 + \frac{1}{r+s} \right) \end{aligned}$$

hence

$$\alpha(p+q) + \beta(r+s) \leq (p+q+l-1) \left(1 + \frac{1}{r+s} \right)$$

and $(\alpha, \beta) \in E_1$.

If $m = 0$, the result derives from the definition of E_0 and $(*)$.

If in $(*)$, $\alpha = 1$, $\beta(r+s) \leq j$.

If in $(*)$, $\alpha = 0$, $\beta(r+s) \leq j + (p+q)$. If moreover $\beta \geq j$, $j(r+s-1) \leq p+q$, hence

- $j = 0$ and $\beta(r+s) \leq p+q$ which is impossible because $\alpha = 0$,
- $j = 1$, and $\beta = 1$.

3) Let $(0, j) \in E_m$, with $m \geq 0$ is minimal. We have

$$j(r+s) \leq (p+q+l-1) \left(1 + \frac{1}{r+s} + \cdots + \frac{1}{(r+s)^m} \right).$$

If $j(r+s) = j+p+q+l-1$, then

$$j \leq (p+q+l-1) \left(\frac{1}{r+s} + \cdots + \frac{1}{(r+s)^m} \right)$$

hence

$$j(r+s) \leq (p+q+l-1) \left(1 + \frac{1}{r+s} + \cdots + \frac{1}{(r+s)^{m-1}} \right).$$

and $(0, j) \in E_{m-1}$ which is contradictory.

4) Let $(i, j) \in E_m$ with $i \geq 2$ and consider part of bidegree $(i, j+p+q+l-1)$. By 1), left member of (I) gives no contribution, we show now that

$$\langle b_k C^k z_2^k (1 + \mu(z))^k \rangle_{i, j+p+q+l-1} = 0.$$

In fact, the monomials which contain z_1^i contain z_2 at the power at least $k+i(l-1)$ with $p+q \leq k \leq p+q+l-1$ and it is sufficient to show that

$$j+p+q+l-1 < k+i(l-1).$$

Moreover, $k \geq p+q$ and $i \geq 2$, hence it is sufficient to prove that $j+p+q+l-1 < p+q+2(l-1)$, i.e.

$$(\spadesuit) \quad j < l-1.$$

By assumption, $(i, j) \in E_m$, therefore

$$j \leq \frac{1}{r+s} (p+q+l-1) \left(1 + \cdots + \frac{1}{(r+s)^m} \right) - i \frac{p+q}{r+s}$$

and condition (\spadesuit) is satisfied if

$$\frac{1}{r+s} (p+q+l-1) \left(1 + \cdots + \frac{1}{(r+s)^m} \right) < (l-1) + 2 \frac{p+q}{r+s}$$

which is clearly satisfied since $r+s \geq 2$. Finally we obtain

$$0 = \lambda A_{ij} C^{p+q+l-1} z_1^i z_2^{j+p+q+l-1}$$

and $A_{ij} = 0$.

5) If $(0, j) \in E_0$, i.e. $j(r+s) \leq p+q+l-1$ then

$$\alpha(p+q) + \beta(r+s) = j + (p+q+l-1) > p+q+l-1$$

hence $(\alpha, \beta) \notin E_0$. If $(0, j) \in E_m \setminus E_{m-1}$,

$$\begin{aligned} (p+q+l-1) \left(1 + \frac{1}{r+s} + \cdots + \frac{1}{(r+s)^{m-1}} \right) &< j(r+s) \\ &\leq (p+q+l-1) \left(1 + \frac{1}{r+s} + \cdots + \frac{1}{(r+s)^m} \right) \end{aligned}$$

and by (*), the new (α, β) satisfies

$$\alpha(p+q) + \beta(r+s) = j + p + q + l - 1$$

We want to check that

$$\alpha(p+q) + \beta(r+s) > (p+q+l-1) \left(1 + \frac{1}{r+s} + \cdots + \frac{1}{(r+s)^m} \right)$$

This is equivalent to

$$j > \frac{1}{r+s} \left(1 + \cdots + \frac{1}{(r+s)^{m-1}} \right) (p+q+l-1)$$

or

$$j(r+s) > \left(1 + \cdots + \frac{1}{(r+s)^{m-1}} \right) (p+q+l-1),$$

i.e. $(0, j) \notin E_{m-1}$ which is true by assumption.

If $k = (p+q) + \beta(r+s) = \beta'(r+s)$ then $p+q$ is a multiple of $r+s$ which is impossible, therefore we have the unicity of (α, β) . \square

Lemma 5. 54 *The linear system with coefficients in $\mathbb{C}[a_1, \dots, a_{l-1}]$ and unknowns*

$$b_{p+q+1}, \dots, b_{p+q+l-1} \quad \text{and} \quad A_{ij}, \quad (i, j) \in E_\infty$$

is a Cramer system of order $l-1 + \text{Card}(E_\infty)$. More precisely, modulo \mathfrak{M} , its determinant is

$$\Delta = C^{p+q+1} \dots C^{p+q+l-1} (\lambda C^{p+q+l-1})^{\text{Card } E_\infty} \neq 0 \pmod{\mathfrak{M}}$$

and $b_k = \frac{B_k}{\Delta}$, $k = p+q+1, \dots, p+q+l-1$, $A_{ij} = \frac{B_{ij}}{\Delta}$, $(i, j) \in E_\infty$, with $B_k, B_{ij} \in \mathbb{C}[a_1, \dots, a_{l-1}]$.

Proof: We order the unknowns in the following way: First unknowns $b_{p+q+1}, \dots, b_{p+q+l-1}$, after coefficients $A_{0j} \neq 0$, with $(0, j) \in E_0$ then $(0, j) \in E_1 \setminus E_0, \dots, (0, j) \in E_{m+1} \setminus E_m$, exhausting E_∞ . Finally coefficients A_{1j} , with j in the decreasing order. We have the same number of equations and of unknowns, therefore we have a linear system of order $l-1 + \text{Card}(E_\infty)$. Let $\mathfrak{M} = (a_1, \dots, a_{l-1})$. In order to prove that we have a Cramer system it is sufficient to prove that modulo \mathfrak{M} the determinant Δ is nonzero. Therefore we consider the equation (I) modulo \mathfrak{M} , i.e.

$$(I_{\mathfrak{M}}) \quad \left\{ \begin{array}{l} \varphi_1 \left(a_0^p \left\{ 1 + \frac{z_1 z_2^{l-1}}{a_0} \right\}^p z_2^{p+q}, a_0^r \left\{ 1 + \frac{z_1 z_2^{l-1}}{a_0} \right\}^r z_2^{r+s} \right) \\ \\ = \lambda \varphi_1(z) \left(C z_2 (1 + \mu(z)) \right)^{p+q+l-1} + \sum_{k=p+q}^{p+q+l-1} b_k C^k z_2^k (1 + \mu(z))^k \pmod{\mathfrak{M}} \end{array} \right.$$

where, in the infinite product $1 + \mu(z)$,

$$\begin{aligned} G(z_1, z_2) &= \left((z_1 z_2^l + a_0 z_2)^p z_2^q, (z_1 z_2^l + a_0 z_2)^r z_2^s \right) \\ &= \left(a_0^p \left(1 + \frac{z_1 z_2^{l-1}}{a_0} \right)^p z_2^{p+q}, a_0^r \left(1 + \frac{z_1 z_2^{l-1}}{a_0} \right)^r z_2^{r+s} \right) \pmod{\mathfrak{M}} \end{aligned}$$

which provides

$$\begin{aligned}
1 + \mu(z) &= \left\{ 1 + \frac{z_1 z_2^{l-1}}{a_0} \right\}^{\frac{r}{r+s}} \left\{ 1 + \frac{a_0^{p+r(l-1)} \left(1 + \frac{z_1 z_2^{l-1}}{a_0} \right)^{p+r(l-1)} z_2^{p+q+(r+s)(l-1)}}{a_0} \right\}^{\frac{r}{(r+s)^2}} \dots \\
&= 1 + \frac{r}{r+s} \frac{z_1 z_2^{l-1}}{a_0} + \frac{r a_0^{p+r(l-1)-1}}{(r+s)^2} z_2^{p+q+(r+s)(l-1)} \\
&\quad + \frac{r(p+r(l-1)) a_0^{p+r(l-1)-2}}{(r+s)^2} z_1 z_2^{p+q+(r+s+1)(l-1)} + \dots \mod \mathfrak{M}
\end{aligned}$$

By construction, the diagonal of the matrix is

$$C^{p+q+1}, \dots, C^{p+q+l-1}, \lambda C^{p+q+l-1}, \dots, \lambda C^{p+q+l-1}$$

and the square submatrix, of order $l-1$ corresponding to the unknowns

$$b_{p+q+i}, \quad i = 1, \dots, l-1,$$

is diagonal because $p+q+(r+s)(l-1) > p+q+l-1$ and no term comes from $(1+\mu(z))$. We shall show that after some linear combinations of the lines, we obtain an upper triangular matrix, which yields $\Delta \neq 0$.

Let $(0, j) \in E_m \setminus E_{m-1}$ ($E_{-1} := \emptyset$). Since $A_{0,j} \neq 0$, the homogeneous part of bidegree $(0, j+p+q+l-1)$ is by lemma 53, 5)

$$A_{\alpha\beta} a_0^{p\alpha+r\beta} z_2^{\alpha(p+q)+\beta(r+s)} = \lambda A_{0j} z_2^j (C z_2)^{p+q+l-1}, \mod \mathfrak{M}$$

with $(\alpha, \beta) \in E_{m+1} \setminus E_m$, if such (α, β) exists, or

$$0 = \lambda A_{0j} z_2^j (C z_2)^{p+q+l-1}, \mod \mathfrak{M}$$

otherwise. A term $b_i z_2^i (1+\mu(z))^i$ has no part of homogeneous bidegree $(0, m)$ because $j+p+q+l-1 < 2(p+q)+(r+s)(l-1)$. Therefore, with the chosen order on the unknowns, all coefficients of the linear equation are over the diagonal of the matrix.

Remain homogeneous parts of bidegree $(1, j+p+q+l-1)$ involving $A_{1,j}$ for $j \geq 1$. We have

$$\begin{aligned}
(1 + \mu(z))^i &= 1 + \frac{ir}{r+s} \frac{z_1 z_2^{l-1}}{a_0} + \frac{ir a_0^{p+r(l-1)-1}}{(r+s)^2} z_2^{p+q+(r+s)(l-1)} \\
&\quad + \frac{ir(p+r(l-1)) a_0^{p+r(l-1)-2}}{(r+s)^2} z_1 z_2^{p+q+(r+s+1)(l-1)} + \dots \mod \mathfrak{M}
\end{aligned}$$

It is easy to check that for $i \geq p+q$,

$$i + p + q + (r+s+1)(l-1) > j + (p+q+l-1),$$

therefore the only terms which may be involved in homogeneous part of bidegree $(1, j+p+q+l-1)$ are

$$b_i C^i z_2^i \frac{ir}{r+s} \frac{z_1 z_2^{l-1}}{a_0}, \quad \text{where } p+q \leq i \leq p+q+l-1$$

therefore

$$i = j + p + q.$$

We have still to check that $j \leq l - 1$. If $(1, j) \in E_0$, then clearly $j \leq l - 1$; if it is not the case, $(1, j) \in E_1 \setminus E_0$ by lemma 53, 2). We have

$$(*) \quad l - 1 < j(r + s) \leq \frac{p + q}{r + s} + (l - 1) \left(1 + \frac{1}{r + s} \right)$$

and $l \geq 2$. With $(*)$ and $p + q \leq r + s - 1$ we check that the inequality $j \leq l - 1$ is still fulfilled.

Now, there are two possibilities

1. There is no (α, β) such that $\alpha(p + q) + \beta(r + s) = j + p + q$. Therefore

$$0 = \lambda A_{1j} z_1 z_2^j (C z_2)^{p+q+l-1} + b_{p+q+j} C^{p+q+j} z_2^{p+q+j} \frac{(p+q+j)r}{r+s} \frac{z_1 z_2^{l-1}}{a_0} \mod \mathfrak{M}$$

The j -th equation (which gives the j -th line L_j of the matrix) is

$$0 = b_{p+q+j} C^{p+q+j} \mod \mathfrak{M}$$

therefore substrating $\frac{(p+q+j)r}{a_0(r+s)} L_j$ we remove the coefficient $b_{p+q+j} C^{p+q+j} \frac{(p+q+j)r}{a_0(r+s)}$ which was under the diagonal.

2. There exists (α, β) such that $\alpha(p + q) + \beta(r + s) = p + q + j$. By lemma 53, 4), there is at most two such coefficients $(0, \beta)$ and $(1, \beta')$. By the choice of the ordering, and lemma 53, 2), $A_{1, \beta'} > A_{1j}$ and the coefficient of $A_{1, \beta'}$

$$-a_0^{p+r\beta'-1} (p + r\beta')$$

is over the diagonal.

Then mod \mathfrak{M} , the homogeneous part of bidegree $(1, j + p + q + l - 1)$ is

$$\begin{aligned} A_{0\beta} a_0^{r\beta} r\beta \frac{z_1 z_2^{l-1}}{a_0} z_2^{\beta(r+s)} + A_{1\beta'} a_0^{p+r\beta'} (p + r\beta') \frac{z_1 z_2^{l-1}}{a_0} z_2^{(p+q)+\beta'(r+s)} \\ = \lambda A_{1j} z_1 z_2^j (C z_2)^{p+q+l-1} + b_{p+q+j} C^{p+q+j} z_2^{p+q+j} \frac{(p+q+j)r}{r+s} \frac{z_1 z_2^{l-1}}{a_0} \end{aligned}$$

hence

$$A_{0\beta} a_0^{r\beta-1} r\beta + A_{1\beta'} a_0^{p+r\beta'-1} (p + r\beta') = \lambda A_{1j} C^{p+q+l-1} + b_{p+q+j} C^{p+q+j} \frac{(p+q+j)r}{r+s} \frac{1}{a_0}$$

where perhaps one of the coefficients $A_{0\beta} = 0$ or $A_{1\beta'} = 0$. The j -th equation derived from the homogeneous part of bidegree $(0, p + q + j)$ is

$$A_{0\beta} a_0^{r\beta} + A_{1\beta'} a_0^{p+r\beta'} = b_{p+q+j} C^{p+q+j} \mod \mathfrak{M}$$

and if $A_{0\beta} = 0$, it remains to subtract $\frac{(p+q+j)r}{a_0(r+s)} L_j$ to obtain a triangular matrix. If $A_{0\beta} \neq 0$, we have two coefficients under the diagonal: $A_{0\beta}$ and b_{p+q+j} . However

$$r\beta = \frac{(p+q+j)r}{r+s}$$

therefore substrating $\frac{(p+q+j)r}{a_0(r+s)} L_j = \frac{r\beta}{a_0} L_j$ we remove both coefficients, obtaining the desired upper triangular matrix.

We conclude that $\Delta = C^{p+q+1} \dots C^{p+q+l-1} (\lambda C^{p+q+l-1})^{\text{Card } E_\infty} \neq 0$. The second member of the Cramer system is nonzero and involves A_{10} and $b_{p+q} = 1$, therefore solutions of the system are rational fractions in variables a_1, \dots, a_{l-1} . \square

Consider the restriction of the equivalence relation defined by L . Since a_0 is fixed we have the extra condition $A = B$, hence by lemma 45,

$$a = (a_1, \dots, a_{l-1}) \sim a' = (a'_1, \dots, a'_{l-1}) \iff a'_i = B^i a_i, \quad \text{for } i = 1, \dots, l-1, l+K,$$

where

$$B^{k-1} = B^{r+s-1} = 1$$

and

$$B^\sigma = B^{p+q+l-1} = B^{p+q+(r+s)l-1} = 1.$$

Let $\Pi_L : \mathbb{C}^{l-1} \rightarrow \mathbb{C}^{l-1}/L$ be the canonical mapping. Similarly, consider the restriction to \mathbb{C}^{l-1} of the equivalence relation of Favre germs given by lemma 23. If we fix λ then $\epsilon^{p+q+l-1} = \epsilon^\sigma = 1$ and

$$b = (b_{p+q+1}, \dots, b_{p+q+l-1}) \sim b' = (b'_{p+q+1}, \dots, b'_{p+q+l-1}) \iff b'_{p+q+i} = \epsilon^i b_{p+q+i}, 1 \leq i \leq l-1.$$

Let $\Pi : \mathbb{C}^{l-1} \rightarrow \mathbb{C}^{l-1}/\mathbb{Z}_{r+s-1}$ the corresponding canonical mapping. We see that the equivalence relations $a \sim a'$ and $b \sim b'$ on \mathbb{C}^{l-1} are equal and $\Pi_L = \Pi$.

Lemma 5.55 *We choose $a_0 \in \mathbb{C}^\star$ and ϵ such that $\epsilon^{r+s-1} = 1$. Let $\sigma = p+q+l-1$ and suppose that $r+s-1$ does not divide $l-d$. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{C}^{l-1} & \xrightarrow{f_{a_0, \epsilon}} & \mathbb{C}^{l-1} \\ \Pi_L \downarrow & & \downarrow \Pi \\ \mathbb{C}^{l-1}/L & \xrightarrow{Id} & \mathbb{C}^{l-1}/\mathbb{Z}_{r+s-1} \end{array}$$

where

$$f_{a_0, \epsilon} : \mathbb{C}^{l-1} \rightarrow \mathbb{C}^{l-1}$$

is an isomorphic polynomial mapping.

Proof: Both canonical mappings

$$\Pi_L : \mathbb{C}^\star \times \mathbb{C}^{l-1} \rightarrow \mathbb{C}^\star \times \mathbb{C}^{l-1}/L \quad \text{and} \quad \Pi : \mathbb{C}^\star \times \mathbb{C}^{l-1} \rightarrow \mathbb{C}^\star \times \mathbb{C}^{l-1}/\mathbb{Z}_{r+s-1}$$

are ramified covering with $r+s-1$ sheets. Let $b = f_{a_0, \epsilon}(a)$ and $b' = f_{a_0, \epsilon}(a')$. By definition of $f_{a_0, \epsilon}$, $G_a \sim F_b$ and $G_{a'} \sim F_{b'}$, therefore

$$a \sim a' \iff G_a \sim G_{a'} \iff F_b \sim F_{b'} \iff b \sim b'$$

and the diagram is commutative. The rational mapping $f_{a_0, \epsilon}$ is proper because if $K \subset \mathbb{C}^{l-1}$ is compact then

$$f_{a_0, \epsilon}^{-1}(K) \subset f_{a_0, \epsilon}^{-1}(\Pi^{-1}(\Pi(K))) = \Pi_L^{-1}(\Pi(K))$$

and $f_{a_0, \epsilon}^{-1}(K)$ is compact as $\Pi_L^{-1}(\Pi(K))$. The rational mapping $f_{a_0, \epsilon}$ is proper hence has no polar set hence is polynomial. The image of $f_{a_0, \epsilon}$ contains an open set by corollary 3.16 therefore $f_{a_0, \epsilon}$ is surjective. The ramified coverings have the same number of sheets hence $f_{a_0, \epsilon}$ is also injective. \square

Lemma 5. 56 We choose $a_0 \in \mathbb{C}^\star$ and ϵ such that $\epsilon^{r+s-1} = 1$. Let $\sigma = p + q + l - 1$ and suppose that $r + s - 1$ does not divide $l - d$. Then there is a proper surjective polynomial mapping

$$\begin{aligned} f_{a_0, \epsilon} : \quad \mathbb{C}^{l-1} &\longrightarrow \mathbb{C}^{l-1} \\ a = (a_1, \dots, a_{l-1}) &\longmapsto (b_{p+q+1}(a), \dots, b_{p+q+l-1}(a)) \end{aligned}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} \right)^r z_2^s \right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^\sigma + \sum_{i=p+q}^{\sigma} b_i z_2^i, z_2^{r+s} \right),$$

where λ depends only on a_0 by 5.51.

Proof: Denote by $f_{a_0, \epsilon}$ be the rational mapping of lemma 54.

1) For $1 \leq j \leq l - 1$, the homogeneous part of bidegree $(0, j + p + q)$ is:

$$\begin{aligned} & b_{p+q+j} C^{p+q+j} z_2^{p+q+j} + \sum_{j'=1}^{j-1} b_{p+q+j'} C^{p+q+j'} z_2^{p+q+j'} P_{jj'}(a_1, \dots, a_{j-j'}) z_2^{j-j'} \\ & - \sum_{\substack{(\alpha, \beta) \neq (1, 0) \\ \alpha(p+q) + \beta(r+s) \leq p+q+j}} A_{\alpha\beta} a_0^{\alpha p + \beta r} z_2^{\alpha(p+q) + \beta(r+s)} \left\langle \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^{l-1} a_i z_2^i \right\}^{\alpha p + \beta r} \right\rangle_{(0, p+q+j-\alpha(p+q)-\beta(r+s))} \\ & = A_{10} z_2^{p+q} \left\langle a_0^p \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^{l-1} a_i z_2^i \right\}^p \right\rangle_{(0, j)} - C^{p+q} z_2^{p+q} \left\langle (1 + \mu(z)) \right\rangle_{(0, j)} \end{aligned}$$

After cancellation of z_2^{p+q+j} and recalling that $A_{10} a_0^p = C^{p+q}$, we obtain the j -th equation

$$\begin{aligned} & b_{p+q+j} C^{p+q+j} + \sum_{j'=1}^{j-1} b_{p+q+j'} C^{p+q+j'} P_{jj'}(a_1, \dots, a_{j-j'}) \\ & - \sum_{\substack{(\alpha, \beta) \neq (1, 0) \\ \alpha(p+q) + \beta(r+s) \leq p+q+j}} A_{\alpha\beta} a_0^{\alpha p + \beta r} \frac{\left\langle \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^{l-1} a_i z_2^i \right\}^{\alpha p + \beta r} \right\rangle_{(0, p+q+j-\alpha(p+q)-\beta(r+s))}}{z_2^{p+q+j-\alpha(p+q)-\beta(r+s)}} \\ & = C^{p+q} \left(\frac{\left\langle \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^{l-1} a_i z_2^i \right\}^p \right\rangle_{(0, j)}}{z_2^j} - \left\langle (1 + \mu(z))^{p+q} \right\rangle_{(0, j)} \right) \end{aligned}$$

We show by *decreasing induction* that for $j = 1, \dots, l - 1$,

$$b_{p+q+j} = b_{p+q+j}(a_1, \dots, a_j) \in \mathbb{C}[a_1, \dots, a_j].$$

For $j = l - 1$ there is nothing to prove. Let $j \geq 1$ and suppose that

$$b_{p+q+l-1} \in \mathbb{C}[a_1, \dots, a_{l-1}], \dots, b_{p+q+j+1} \in \mathbb{C}[a_1, \dots, a_{j+1}].$$

Then

- For $j' = 1, \dots, j - 1$, $P_{jj'} \in \mathbb{C}[a_1, \dots, a_{j-1}]$,
- Since $p + q + j - \alpha(p + q) - \beta(r + s) < j$,

$$\frac{\left\langle \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^{l-1} a_i z_2^i \right\}^{\alpha p + \beta r} \right\rangle_{(0, p+q+j-\alpha(p+q)-\beta(r+s))}}{z_2^{p+q+j-\alpha(p+q)-\beta(r+s)}} \in \mathbb{C}[a_1, \dots, a_{j-1}],$$

- Clearly

$$\frac{\left\langle \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^{l-1} a_i z_2^i \right\}^p \right\rangle_{(0, j)}}{z_2^j} = \frac{\left\langle \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^j a_i z_2^i \right\}^p \right\rangle_{(0, j)}}{z_2^j} \in \mathbb{C}[a_1, \dots, a_j]$$

- From the definition of μ ,

$$\frac{\left\langle (1 + \mu(z))^{p+q} \right\rangle_{(0, j)}}{z_2^j} = \frac{\left\langle \left\{ 1 + \frac{1}{a_0} \sum_{i=1}^j a_i z_2^i \right\}^{\frac{r(p+q)}{r+s}} \dots \right\rangle}{z_2^j} \in \mathbb{C}[a_1, \dots, a_j]$$

Therefore modulo $\mathfrak{M}_{j-1} = (a_1, \dots, a_{j-1})$,

$$b_{p+q+j} C^j = \frac{\delta a_j}{(r + s) a_0}$$

and there exists a polynomial $R_j(a_1, \dots, a_{j-1}) \in \mathbb{C}[a_1, \dots, a_{j-1}]$ without constant term such that

$$(T) \quad b_{p+q+j} = b_{p+q+j}(a_1, \dots, a_j) = \frac{\delta a_j}{C^j(r + s) a_0} + R_j(a_1, \dots, a_{j-1}).$$

2) We show now that the polynomial mapping

$$f : \begin{array}{ccc} \mathbb{C}^{l-1} & \longrightarrow & \mathbb{C}^{l-1} \\ (a_1, \dots, a_{l-1}) & \longmapsto & (b_{p+q+1}(a_1), \dots, b_{p+q+j}(a_1, \dots, a_j), \dots, b_{p+q+l-1}(a_1, \dots, a_{l-1})) \end{array}$$

is proper, hence surjective. Let $K \subset \mathbb{C}^{l-1}$ be a compact subset and $(a_1, \dots, a_{l-1}) \in f^{-1}(K)$. We have to show that for any $j = 1, \dots, l - 1$, a_j is uniformly bounded. We show this property by an *increasing induction* on $j \geq 1$. For $j = 1$,

$$(R1) \quad b_{p+q+1} C = \frac{\delta a_1}{(r + s) a_0}$$

and a_1 is bounded since b_{p+q+1} is. Suppose that $j \geq 2$ and that a_1, \dots, a_{j-1} are uniformly bounded. Then a_j is uniformly bounded since it is the cas of $R_j(a_1, \dots, a_{j-1})$ and b_{p+q+j} . \square

B) We suppose that $l - d = K(r + s - 1)$ and $\lambda = 1$ i.e. there are non trivial global vector fields. We have

$$l + K = d + Kk(S), \quad \sigma = p + q + l - 1 = (k(S) - 1)(K + 1), \quad \frac{\sigma k(S)}{k(S) - 1} = k(S)(K + 1).$$

We denote by

$$(\Sigma) := \left(\sum_{i=1}^{l-1} a_i z_2^i + a_{l+K} z_2^{l+K} + z_1 z_2^{l-1} \right)$$

the equation (I) is now

$$(I) \quad \begin{cases} \varphi_1 \left(a_0^p \left\{ 1 + \frac{1}{a_0} (\Sigma) \right\}^p z_2^{p+q}, a_0^r \left\{ 1 + \frac{1}{a_0} (\Sigma) \right\}^r z_2^{r+s} \right) \\ = \lambda \varphi_1(z) \left(C z_2 (1 + \mu(z)) \right)^\sigma + \sum_{i=p+q}^{\sigma} b_i \left(C z_2 (1 + \mu(z)) \right)^i + c \left(C z_2 (1 + \mu(z)) \right)^{\frac{\sigma k}{k-1}} \end{cases}$$

If $m \geq \frac{\sigma}{k-1}$, then $(0, m) \notin E_\infty$, in fact

$$m(r+s) \geq \frac{\sigma k}{k-1} = (p+q+l-1) \frac{r+s}{r+s-1},$$

therefore the coefficients a_{l+K} and c doesn't occur in the previous calculations and we obtain by similar arguments a polynomial mapping $f_{a_0, \epsilon}$.

Lemma 5. 57 Suppose $l = d + K(r+s-1)$. Let $\mathfrak{M} = (a_1, \dots, a_{l-1})$ and (i, j) such that

$$\left\langle A_{ij} a_0^{pi+rj} \left\{ 1 + \frac{1}{a_0} \left(\sum_{m=1}^{l-1} a_m z_2^m + a_{l+K} z_2^{l+K} + z_1 z_2^{l-1} \right) \right\}^{pi+rj} z_2^{i(p+q)+j(r+s)} \right\rangle_{(0, \frac{\sigma k}{k-1}) \mod \mathfrak{M}} \neq 0,$$

then, $(i, j) = (1, 0)$ or $(i, j) = (0, \frac{\sigma}{k-1})$. More precisely homogeneous part of bidegree $(0, \frac{\sigma k}{k-1})$ is

$$c C^{\frac{\sigma k}{k-1}} = A_{10} p a_0^{p-1} a_{l+K} + A_{0, \frac{\sigma}{k-1}} C^\sigma (1 - \lambda) \mod \mathfrak{M}.$$

In particular if there are global vector fields, i.e. $\lambda = 1$,

$$c C^{\frac{\sigma k}{k-1}} = A_{10} p a_0^{p-1} a_{l+K} \mod \mathfrak{M}.$$

Proof: 1) If $(i, j) \in E_\infty$, then $i(p+q) + j(r+s) < \frac{\sigma k}{k-1}$ and $i \leq 1$.

- Case $i = 1$: Since

$$l + K + (p+q) + j(r+s) = \frac{\sigma k}{k-1} + jk \geq \frac{\sigma k}{k-1}$$

we have equality if $j = 0$ hence $(i, j) = (1, 0)$.

- Case $i = 0$: then $1 \leq j \leq \frac{\sigma}{k-1}$ and mod \mathfrak{M} ,

$$\left\langle A_{0j} a_0^{rj} \left\{ 1 + \frac{a_{l+K} z_2^{l+K}}{a_0} \right\}^{rj} z_2^{j(r+s)} \right\rangle_{(0, \frac{\sigma k}{k-1}) \mod \mathfrak{M}} \neq 0, \mod \mathfrak{M}$$

In the left member the possible powers of z_2 are of the form $\alpha(l+K) + jk$ with $\alpha \geq 0$ and $j \geq 1$ such that

$$\alpha(l+K) + jk = \frac{\sigma k}{k-1}.$$

Since $\alpha(l+K) + jk = (d+Kk)\alpha + jk$, we derive that $\alpha \geq 1$ is impossible, therefore $\alpha = 0$ and $j = \frac{\sigma}{k-1}$.

2) From 1) we deduce that there exists a polynomial P in variables a_1, \dots, a_{l-1} such that the coefficients of $z_2^{\frac{\sigma k}{k-1}}$ in (I) give the equality

$$A_{10} p a_0^{p-1} a_{l+K} + A_{0, \frac{\sigma}{k-1}} a_0^{\frac{\sigma r}{k-1}} = \lambda A_{0, \frac{\sigma}{k-1}} C^\sigma + c C^{\frac{\sigma k}{k-1}} + P(a_1, \dots, a_{l-1}).$$

By equation (5),

$$a_0^{\frac{\sigma r}{k-1}} - \lambda C^\sigma = C^\sigma (\lambda - 1)$$

which gives the result. \square

Lemma 5. 58 *If $l - d = K(r + s - 1)$ and $\lambda = 1$, there is a bijective polynomial mapping*

$$\begin{aligned} g_{a_0, \epsilon} : \quad \mathbb{C}^{l-1} \times \mathbb{C} &\longrightarrow \mathbb{C}^{l-1} \times \mathbb{C} \\ a = (a_1, \dots, a_{l-1}, a_{l+K}) &\longmapsto (b_{p+q+1}(a), \dots, b_{p+q+l-1}(a), c(a)) \end{aligned}$$

such that

$$G(z_1, z_2) = \left(\left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^p z_2^q, \left(z_1 z_2^l + \sum_{i=0}^{l-1} a_i z_2^{i+1} + a_{l+K} z_2^{2l+K} \right)^r z_2^s \right)$$

is conjugated to the polynomial germ

$$F(z_1, z_2) = \left(\lambda z_1 z_2^\sigma + \sum_{k=p+q}^{\sigma} b_k z_2^k + c z_2^{\frac{\sigma k(S)}{k(S)-1}}, z_2^{r+s} \right).$$

Proof: We have a bijective polynomial map

$$f_{a_0, \epsilon} : \mathbb{C}^{l-1} \rightarrow \mathbb{C}^{l-1}, \quad a \mapsto b = f_{a_0, \epsilon}(a).$$

From lemma 5.57, when $a = (a_1, \dots, a_{l-1})$ is fixed and $a_{l+K} \in \mathbb{C}$, the mapping $c : \mathbb{C} \rightarrow \mathbb{C}$, $a_{l+K} \mapsto c = c(a_{l+K}) = C^{-\frac{\sigma k}{k-1}} A_{10} p a_0^{p-1} a_{l+K}$ is linear hence bijective. \square

Corollary 5. 59 *Any surface with GSS with one tree admits a special birational structure.*

Corollary 5. 60 *The intersection $A := \text{Aut}(\mathbb{C}^2, H, 0) \cap \Phi$ is the trivial group or a group isomorphic to $(\mathbb{C}, +)$. Moreover*

- if $k - 1$ does not divide $\mathfrak{s} = p + q + l - 1$, the canonical mapping

$$g : \mathcal{G}/A = \mathcal{G}(p, q, r, s, l)/A \rightarrow U_{k, \mathfrak{s}, m_1}/\mathbb{Z}_{k-1}$$

to the Oeljeklaus-Toma coarse moduli space of marked surfaces (S, C_0) with one tree

$$U_{k, \mathfrak{s}, m_1}/\mathbb{Z}_{k-1} = \mathbb{C}^\star \times \mathbb{C}^{l-1}/\mathbb{Z}_{k-1}$$

is isomorphic and there is a polynomial lifting

$$(\lambda, b) : \mathbb{C}^\star \times \mathbb{C}^{l-1} \rightarrow \mathbb{C}^\star \times \mathbb{C}^{l-1}$$

which is a covering such that

$$\begin{array}{ccc} \mathbb{C}^\star \times \mathbb{C}^{l-1} & \xrightarrow{(\lambda, b)} & \mathbb{C}^\star \times \mathbb{C}^{l-1} \\ \downarrow & & \downarrow \\ \mathcal{G}/A & \xrightarrow{g} & U_{k, \mathfrak{s}, m_1}/\mathbb{Z}_{k-1} \end{array}$$

is commutative,

- if $k - 1$ divides $\mathfrak{s} = p + q + l - 1$, we have similar results for

$$U_{k,\mathfrak{s},m_1}^{\lambda \neq 0, c=0} / \mathbb{Z}_{k-1} \quad \text{and} \quad U_{k,\mathfrak{s},m_1}^{\lambda=1} / \mathbb{Z}_{k-1}.$$

Corollary 5. 61 *Let $\mathcal{S}_{J,\sigma} \rightarrow B_J$ be a large family with $\sigma = Id$. Let $T_{J,\sigma}$ the hypersurface where cocycles $[\theta^i]$ and $[\mu^i]$ are not independent. Then for each stratum $B_{J,M}$, the trace $T_{J,\sigma} \cap B_{J,M}$ on $B_{J,M}$ is equal to the inverse image of the ramification set by the lift of the canonical mapping i.e.*

- If $k - 1$ does not divide \mathfrak{s} ,

$$T_{J,\sigma} \cap B_{J,M} = (\lambda, b)^{-1}(T_{k,\mathfrak{s},m_1}),$$

- If $k - 1$ divides \mathfrak{s}

$$T_{J,\sigma} \cap B_{J,M} = (\lambda, b)^{-1}(T_{k,\mathfrak{s},m_1}^{\lambda \neq 1, c=0}).$$

In particular in B_J there is no curve over which the surfaces are isomorphic.

6 Appendix

6.1 On logarithmic deformations of surfaces with GSS, by Laurent Bruasse

The results contained in this section is a not yet published part of the thesis [3]. Notations are those of [4] and [9].

Let \mathcal{F} be a reduced foliation on a compact complex surface S . We denote by $T_{\mathcal{F}}$ (resp. $N_{\mathcal{F}}$) the tangent (resp. normal) line bundle to \mathcal{F} .

Let p be a singular point of the foliation; in a neighbourhood of p endowed with a coordinate system (z, w) in which $p = (0, 0)$, \mathcal{F} is defined by a holomorphic vector field

$$\theta(z, w) = A(z, w) \frac{\partial}{\partial z} + B(z, w) \frac{\partial}{\partial w}.$$

Let $J(z, w)$ be the jacobian matrix of the mapping (A, B) . Baum-Bott [1] and Brunella [4] have introduced the following two indices:

$$Det(p, \mathcal{F}) = Res_{(0,0)} \frac{\det J(z, w)}{A(z, w)B(z, w)} dz \wedge dw$$

$$Tr(p, \mathcal{F}) = Res_{(0,0)} \frac{(\text{tr } J(z, w))^2}{A(z, w)B(z, w)} dz \wedge dw$$

where $Res_{(0,0)}$ is the residue at $(0, 0)$ (see [15] p649). We denote by $S(\mathcal{F})$ the singular set of \mathcal{F} , it is a finite set of points, and let

$$Det \mathcal{F} := \sum_{p \in S(\mathcal{F})} Det(p, \mathcal{F}), \quad Tr(\mathcal{F}) := \sum_{p \in S(\mathcal{F})} Tr(p, \mathcal{F}).$$

Proposition 6. 62 (Baum-Bott formulas, [1],[4]) *We have*

$$Det \mathcal{F} = c_2(S) - c_1(T_{\mathcal{F}}).c_1(S) + c_1(T_{\mathcal{F}})^2,$$

$$Tr(\mathcal{F}) = c_1(S)^2 - 2c_1(T_{\mathcal{F}}).c_1(S) + c_1(T_{\mathcal{F}})^2.$$

By [9], if S is a minimal compact complex surface with GSS, then

$$\text{Det } \mathcal{F} = n, \quad \text{Tr}(\mathcal{F}) = 2n - \sigma_n(S).$$

Proposition 6. 63 *Let S be a minimal surface containing a GSS with $n = b_2(S) \geq 1$ and $\text{tr}(S) = 0$. If \mathcal{F} is a reduced foliation on S , then*

$$h^1(S, T_{\mathcal{F}}) = \begin{cases} 3n - \sigma_n(S) & \text{if } h^0(S, \Theta) = 0 \\ 3n - \sigma_n(S) + 1 & \text{if } h^0(S, \Theta) = 1 \end{cases}$$

If there is no non-trivial global vector fields this integer is precisely the number of generic blowing-ups.

Proof: By Riemann-Roch formula

$$\begin{aligned} h^0(T_{\mathcal{F}}) - h^1(T_{\mathcal{F}}) + h^2(T_{\mathcal{F}}) &= \chi(S) + \frac{1}{2} \left(c_1(T_{\mathcal{F}})^2 - c_1(T_{\mathcal{F}})c_1(K) \right) \\ &= c_1(T_{\mathcal{F}})^2 = \sigma_n(S) - 3n \end{aligned}$$

since

- by the first Baum-Bott formula $c_1(T_{\mathcal{F}}).c_1(S) = c_1(T_{\mathcal{F}})^2$ and
- by the second and the previous observation $c_1(T_{\mathcal{F}})^2 = -\text{Tr}(\mathcal{F}) + c_1(S)^2 = \sigma_n(S) - 3n$.

Suppose first that S is of intermediate type. Two cases occur

$$h^0(T_{\mathcal{F}}) = \begin{cases} 0 & \text{if } h^0(S, \Theta) = 0 \\ 1 & \text{if not} \end{cases}$$

Moreover, by Serre duality $h^2(T_{\mathcal{F}}) = h^0(K \otimes T_{\mathcal{F}}^*)$ and

$$c_1(K).(c_1(K) - c_1(T_{\mathcal{F}})) = c_1(S)^2 + c_1(S).c_1(T_{\mathcal{F}}) = -n + (\sigma_n(S) - 3n) = -4n + \sigma_n(S) < 0$$

Let e_i , $i = 0, \dots, n-1$ be the Donaldson classes in $H^2(S, \mathbb{Z})$ which trivialize the negative intersection form. In $H^2(S, \mathbb{Z})$, $c_1(K) = \sum_{i=0}^{n-1} e_i$ and $c_1(K) - c_1(T_{\mathcal{F}}) = \sum_{i=0}^{n-1} a_i e_i$. Since $c_1(K).(c_1(K) - c_1(T_{\mathcal{F}})) < 0$ we have $\sum_i a_i > 0$ therefore $h^0(K \otimes T_{\mathcal{F}}^*) = 0$ by [24] Lemma (2.3). If S is a Inoue-Hirzebruch surface, there are two foliations, each defined by a twisted vector field $\theta \in H^0(S, \Theta \otimes L^\lambda)$, with λ an irrational quadratic number (see [9]), hence $T_{\mathcal{F}} = L^{1/\lambda}$. We have $-K = D$ or $-K^{\otimes 2} = 2D$ and there is no topologically trivial divisor, therefore

$$h^0(L^{1/\lambda}) = 0, \quad \text{and} \quad h^2(L^{1/\lambda}) = h^0(K \otimes L^\lambda) = 0.$$

We conclude by Riemann-Roch theorem that

$$h^1(T_{\mathcal{F}}) = h^1(L^{1/\lambda}) = 0 = 3n - \sigma_n(S)$$

which is the annouced result. □

We have a canonical injection $0 \xrightarrow{i} T_{\mathcal{F}} \rightarrow \Theta(-\text{Log } D)$. The aim of the following proposition is to compare logarithmic deformations and deformations which respect the foliation:

Proposition 6. 64 *There exists an exact sequence of sheaves of \mathcal{O}_S -Modules*

$$(\spadesuit) \quad 0 \rightarrow T_{\mathcal{F}} \xrightarrow{i} \Theta(-\text{Log } D) \rightarrow N_{\mathcal{F}} \otimes \mathcal{O}(-D) \rightarrow 0.$$

Proof: Let $\mathcal{U} = (U_i)$ be a finite covering by open sets endowed with holomorphic 1-forms ω_i defining the foliation \mathcal{F} . On each open set U_i we consider the morphism

$$\begin{array}{ccc} j : \Theta(-\text{Log } D)|_{U_i} & \longrightarrow & N_{\mathcal{F}} \otimes \mathcal{O}_S(-D)|_{U_i} \\ \theta & \longmapsto & \omega_i(\theta) \end{array}$$

Since θ is tangent to D , $\omega_i(\theta)$ vanishes on D , therefore the morphism is well defined on U_i . Moreover, by definition, the normal bundle $N_{\mathcal{F}}$ is defined by the cocycle $(g_{ij})_{ij} = (\omega_i/\omega_j)_{ij} \in H^1(\mathcal{U}, \mathcal{O}^*)$, therefore j is well defined on S and its kernel is clearly $\text{Im } i$. It remains to check that j is surjective: outside D it is obvious since the foliation has singular points only at the intersection of two curves and we have the exact sequence

$$0 \rightarrow T_{\mathcal{F}} \xrightarrow{i} \Theta(-\text{Log } D)|_{S \setminus D} = \Theta|_{S \setminus D} \rightarrow N_{\mathcal{F}} \rightarrow 0.$$

Let $x \in D$, $f_x \in N_{\mathcal{F},x} \otimes \mathcal{O}(-D)_x$ and U an open neighbourhood of x on which f is defined.

- If x is not at the intersection of two curves, let (z, w) be a coordinate system in which $D = \{z = 0\}$ and \mathcal{F} defined by $\omega = a(z, w)dz + zb(z, w)dw$. Since f vanishes on D , $f = zg$. Let $\theta = z\alpha(z, w)\frac{\partial}{\partial z} + \beta(z, w)\frac{\partial}{\partial w}$ be a logarithmic vector field. We have to find α and β such that

$$f(z, w) = zg(z, w) = \omega(\theta) = z(a\alpha + b\beta)$$

i.e. $g \in (a, b)$. The are solutions because x is not a singular point of the foliation hence a is invertible at x .

- If x is at the intersection of two curves,

$$\omega = wadz + zbdw, \quad f = zwg \quad \text{and} \quad \theta = z\alpha(z, w)\frac{\partial}{\partial z} + w\beta(z, w)\frac{\partial}{\partial w}.$$

We have to solve $g = a\alpha + b\beta$. By [19] p171 (see also [9] p1528), the order of θ is one, hence a or b is invertible and $g \in (a, b)$.

□

Let S be a minimal compact complex surface containing a GSS with $n = b_2(S) \geq 1$ and $\text{tr}(S) = 0$. If S is not a Inoue-Hirzebruch surface then S admits a unique holomorphic foliation \mathcal{F} [9] p1540 given by a d -closed section of $H^0(S, \Omega^1(\text{Log } D) \otimes L^{k(S)})$. If S is a Inoue-Hirzebruch surface, it admits exactly two foliations defined by twisted vector fields.

The exact sequence (♠) yields

$$0 \rightarrow H^1(S, T_{\mathcal{F}}) \rightarrow H^1(S, \Theta(\text{Log } D)) \rightarrow H^1(S, N_{\mathcal{F}} \otimes \mathcal{O}(-D)) \rightarrow H^2(S, T_{\mathcal{F}})$$

In fact, if S is not a Inoue-Hirzebruch surface, \mathcal{F} is unique, thence Ω^1 contains a unique non-trivial coherent subsheaf which is $\mathcal{O}(-D) \otimes L^{1/k}$. As $N_{\mathcal{F}}^*$ is another, $N_{\mathcal{F}} = \mathcal{O}(D) \otimes L^k$ and $H^0(S, N_{\mathcal{F}} \otimes \mathcal{O}(-D)) = H^0(S, L^k) = 0$ because $k \neq 1$. We have also $h^2(S, N_{\mathcal{F}} \otimes \mathcal{O}(-D)) = h^2(S, L^k) = h^0(S, K \otimes L^{1/k}) = 0$. By Riemann-Roch theorem, $h^1(S, N_{\mathcal{F}} \otimes \mathcal{O}(-D)) = 0$ and we obtain the isomorphism

$$0 \rightarrow H^1(S, T_{\mathcal{F}}) \rightarrow H^1(S, \Theta(\text{Log } D)) \rightarrow 0$$

If S is a Inoue-Hirzebruch surface $N_{\mathcal{F}} = \mathcal{O}(D) \otimes L^\lambda$ where λ is an irrational number and we have the same conclusion.

With (63) we have proved:

Theorem 6. 65 *Let S be a minimal compact complex surface containing a GSS with $n = b_2(S) \geq 1$ and $\text{tr}(S) = 0$. Then:*

$$h^1(S, \Theta(\text{Log } D)) = h^1(S, T_{\mathcal{F}}) = \begin{cases} 3n - \sigma_n(S) & \text{if } h^0(S, \Theta) = 0 \\ 3n - \sigma_n(S) + 1 & \text{if } h^0(S, \Theta) = 1 \end{cases}$$

In particular any logarithmic deformation keeps the foliation.

Remark 6. 66 *If $\text{tr}(S) \neq 0$, the theorem remains true by [8].*

6.2 The torsion of some first derived direct image sheaves, by Andrei Teleman[29]

Let B, P be complex manifolds $\pi : P \rightarrow B$ a proper holomorphic submersion and let \mathcal{E} be a holomorphic bundle on P . We are interested in the torsion of the sheaf $R^1\pi_*(\mathcal{E})$.

Let $U \subset B$ be an open set and $\varphi \in \mathcal{O}(U)$ a non-trivial holomorphic function, and $D := Z(\varphi)$ the associated effective divisor. We are interested in the sheaf $\text{Ker}(m_\varphi)$ where $m_\varphi : R^1\pi_*(\mathcal{E})|_U \rightarrow R^1\pi_*(\mathcal{E})|_U$ is the morphism defined by multiplication with φ .

By definition of the \mathcal{O}_B -module structure on $R^1\pi_*(\mathcal{E})$, the morphism m_φ is just $R^1\pi_*(m_\Phi)$, where m_Φ is the morphism of sheaves $\mathcal{E}|_{\pi^{-1}(U)} \rightarrow \mathcal{E}|_{\pi^{-1}(U)}$ defined by multiplication with the function

$$\Phi := \pi^*(\varphi) = \varphi \circ \pi \in \mathcal{O}(\pi^{-1}(U)) .$$

Tensorizing by the locally free sheaf \mathcal{E} the tautological exact sequence associated with the divisor $\Delta = Z(\Phi)$, we obtain the short exact sequence

$$0 \longrightarrow \mathcal{E}|_{\pi^{-1}(U)} \xrightarrow{m_\Phi} \mathcal{E}|_{\pi^{-1}(U)} \longrightarrow \mathcal{E}_\Delta \longrightarrow 0 ,$$

which yields a long exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*(\mathcal{E}|_{\pi^{-1}(U)}) &\xrightarrow{\pi_*(m_\Phi)} \pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \longrightarrow \pi_*(\mathcal{E}_\Delta) \longrightarrow \\ &\longrightarrow R^1\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \xrightarrow{R^1\pi_*(m_\Phi)} R^1\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \rightarrow \cdots \end{aligned} \quad (9)$$

Denote by j and J the inclusions of D and Δ in B and P respectively. The sheaf \mathcal{E}_Δ can be written as $J_*(\mathcal{E}|_\Delta)$. One has $\pi \circ J = j \circ (\pi|_\Delta)$, hence

$$\pi_*(\mathcal{E}_\Delta) = \pi_*(J_*(\mathcal{E}|_\Delta)) = (\pi \circ J)_*(\mathcal{E}|_\Delta) = (j \circ \pi|_\Delta)_*(\mathcal{E}|_\Delta) = j_*[(\pi|_\Delta)_*(\mathcal{E}|_\Delta)] .$$

We consider the Brill-Noether locus

$$BN(\mathcal{E}) := \{x \in B \mid h^0(\mathcal{E}_x) \neq 0\} \subset B .$$

Lemma 6. 67 *Suppose that the divisor $Z(\varphi)$ is reduced, and that $BN(\mathcal{E}) \cap Z(\varphi)$ has codimension ≥ 2 at every point. Then*

$$\text{Ker}(m_\varphi : R^1\pi_*(\mathcal{E})|_U \rightarrow R^1\pi_*(\mathcal{E})|_U) = 0 .$$

Proof: It suffices to prove that $(\pi|_\Delta)_*(\mathcal{E}|_\Delta) = 0$. Let $V \subset D := Z(\varphi)$ be an open set and W its pre-image in Δ . One has

$$(\pi|_\Delta)_*(\mathcal{E}|_\Delta)(V) = H^0(W, \mathcal{E}|_\Delta) .$$

Since W is reduced, the vanishing of a section $s \in H^0(W, \mathcal{E}|_\Delta)$ can be tested pointwise. But the restriction of any such section to the dense set

$$W \setminus (\pi|_\Delta)^{-1}(BN(\mathcal{E}))$$

vanish obviously (because it vanishes fibrewise). This shows $H^0(W, \mathcal{E}|_\Delta) = 0$. \square

Proposition 6. 68 *Suppose that the Brill-Noether locus*

$$BN(\mathcal{E}) := \{x \in B \mid h^0(\mathcal{E}_x) \neq 0\} \subset B$$

has codimension ≥ 2 at every point. Then $R^1\pi_(\mathcal{E})$ is torsion free.*

Proof: It suffices to prove that for every $x \in B$ and for any irreducible germ $\varphi_x \in \mathcal{O}_x$ the multiplication morphism $m_{\varphi_x} : R^1\pi_*(\mathcal{E})_x \rightarrow R^1\pi_*(\mathcal{E})_x$ by φ_x is injective. Choose $U \subset B$ a sufficiently small open neighborhood of x such that φ is defined on U and the effective divisor $Z(\varphi) \subset U$ is reduced¹.

Then

$$\text{Ker}(m_{\varphi_x}) = \{\text{Ker}(m_{\varphi} : R^1\pi_*(\mathcal{E})|_U \rightarrow R^1\pi_*(\mathcal{E})|_U)\}_x = 0$$

by Lemma 67 and the exact sequence (6.2). □

Proposition 6. 69 *Suppose that $B \ni x \mapsto h^0(\mathcal{E}_x) \in \mathbb{N}$ is constant. Then $R^1\pi_*(\mathcal{E})$ is torsion free.*

Proof: Since the map $B \ni x \mapsto h^0(\mathcal{E}_x) \in \mathbb{N}$ is constant, $\pi_*(\mathcal{E})$ is locally free and commutes with base change by Grauert's theorems. Here we used the properness and the flatness of p (which implies the flatness of \mathcal{E} over B). This implies that the natural morphism

$$\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \longrightarrow \pi_*(\mathcal{E}_{\Delta})$$

can be identified with the morphism $\pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \longrightarrow \pi_*(\mathcal{E}|_{\pi^{-1}(U)}) \otimes \mathcal{O}_D$, which is obviously surjective. Therefore $\text{Ker}(m_{\varphi} : R^1\pi_*(\mathcal{E})|_U \rightarrow R^1\pi_*(\mathcal{E})|_U) = \{0\}$ by the exact sequence (6.2). □

Theorem 6. 70 *Suppose that*

1. *The fibers of π are connected surfaces.*
2. *The Brill-Noether locus*

$$BN(\mathcal{E}) := \{x \in B \mid h^0(\mathcal{E}_x) \neq 0\} \subset B$$

has codimension ≥ 2 at every point.

3. *The map $B \ni x \mapsto h^2(\mathcal{E}_x) \in \mathbb{N}$ is constant.*

Then

1. *$R^1\pi_*\mathcal{E}$ is torsion free.*
2. *Let k denote the rank of $R^1\pi_*\mathcal{E}$ and $s = (s_1, \dots, s_k)$ be a system of sections in $H^0(B, R^1\pi_*\mathcal{E})$ such that $s(x)$ is linearly independent in the fiber $R^1\pi_*\mathcal{E}(x)$ for every $x \in B \setminus BN(\mathcal{E})$. Then $s(x)$ is linearly independent in $R^1\pi_*\mathcal{E}(x)$ for every $x \in B$.*

Proof: The first statement follows from Proposition 68. For the second, denote by \mathcal{F} the torsion free sheaf $R^1\pi_*\mathcal{E}$ on B . Since the map $B \ni x \mapsto h^2(\mathcal{E}_x) \in \mathbb{N}$ is constant, it follows by Grauert's theorems that $R^2\pi_*\mathcal{E}$ is locally free and that $R^2\pi_*\mathcal{E}$, $R^1\pi_*\mathcal{E}$ commute with base changes ([2] Theorem 3.4 p. 116). In particular the canonical morphisms $R^i\pi_*\mathcal{E}(x) \rightarrow H^i(\mathcal{E}_x)$ are isomorphisms for $i = 1, 2$, and for every $x \in B$.

By Riemann-Roch theorem and the third assumption it follows that the map $B \ni x \mapsto h^1(\mathcal{E}_x)$ is constant on $B \setminus BN(\mathcal{E})$, and the sheaf $R^1\pi_*\mathcal{E}$ is locally free on this open subset. The system s defines a morphism $\sigma : \mathcal{O}_B^{\oplus k} \rightarrow \mathcal{F}$, which is a bundle isomorphism on $B \setminus BN(\mathcal{E})$. We will show that $\sigma(x) : \mathbb{C}^k \rightarrow \mathcal{F}(x_0)$ is injective for any $x_0 \in BN(\mathcal{E})$. Let x_0 be such a point and $S \subset B$ be smooth locally closed surface such that $S \cap BN(\mathcal{E}) = \{x_0\}$, let $\pi^S : P^S := \pi^{-1}(S) \rightarrow S$

¹Being reduced at a point is an open property. Indeed the set of points of a complex space X at which X is reduced coincide with the complement of the support of the ideal sheaf of nilpotents of the structure sheaf \mathcal{O}_X . On the other hand $Z(\varphi)$ is reduced at x because it is irreducible at this point. Note that being irreducible at a point is not an open property in complex analytic geometry.

the restricted fibrations, and $\mathcal{E}^S := \mathcal{E}|_{P^S}$. Recalling that $R^1\pi_*$ commutes with base changes we put

$$\mathcal{F}^S := R^1\pi_*^S \mathcal{E}^S = \mathcal{F}|_S .$$

It suffices to prove that the restriction $\sigma_S : \mathcal{O}_S^{\oplus k} \rightarrow \mathcal{F}^S$ induces a monomorphism $\mathbb{C}^k \rightarrow \mathcal{F}^S(x_0) = \mathcal{F}(x_0)$. By Lemma 71 below, it suffices to prove that the induced morphism $\sigma_S^\vee : [\mathcal{F}^S]_{x_0}^\vee \rightarrow [\mathcal{O}_{x_0, S}^{\oplus k}]^\vee$ is surjective. Since S is a smooth surface and $[\mathcal{F}^S]^\vee$ is reflexive on S , it will also be free by [20] Corollary 5.20. The morphism $[\mathcal{F}^S]^\vee \rightarrow [\mathcal{O}_S^{\oplus k}]^\vee$ is just a morphism of rank k locally free sheaves on S ; $\wedge^k(\sigma_S^\vee)$ is an isomorphism on $S \setminus \{x_0\}$, so it will be an isomorphism everywhere on S . Therefore σ_S^\vee is an isomorphism on S . \square

Lemma 6. 71 *Let (A, \mathfrak{m}) be a local ring with residual field K , and $f : U \rightarrow V$ an A -module morphism, where U is free and finitely generated. If the morphism $f^\vee : V^\vee \rightarrow U^\vee$ is surjective, then the induced vector space morphism $\phi : K^m \simeq U \otimes_A K \rightarrow V \otimes_A K$ is injective.*

Proof: Let $x \in \text{Ker } \phi$, and let u be a lift of x in U . The condition $\phi(x) = 0$ becomes $f(u) \in \mathfrak{m}V$. Since f^\vee is surjective, one obtains for every $\mathfrak{u} \in U^\vee$ an element $\mathfrak{v} \in V^\vee$ such that $f^\vee(\mathfrak{v}) = \mathfrak{u}$, so

$$\langle \mathfrak{u}, u \rangle = \langle f^\vee(\mathfrak{v}), u \rangle = \langle \mathfrak{v}, f(u) \rangle \in \mathfrak{m} ,$$

In particular the components u_i of u with respect to a basis in U belong all to \mathfrak{m} , so $u \in \mathfrak{m}U$. \square

Corollary 6. 72 *Suppose that*

1. *The fibers of π are connected surfaces.*
2. *The Brill-Noether locus*

$$BN(\mathcal{E}) := \{x \in B \mid h^0(\mathcal{E}_x) \neq 0\} \subset B$$

has codimension ≥ 2 at every point.

3. *The map $B \ni x \mapsto h^2(\mathcal{E}_x) \in \mathbb{N}$ is constant*
4. *The rank of the coherent sheaf $R^1\pi_*\mathcal{E}$ is k and there exists a system of global sections $s = (s_1, \dots, s_k)$ in $H^0(B, R^1\pi_*\mathcal{E})$ such that $s(x)$ is linearly independent in the fiber $R^1\pi_*\mathcal{E}(x)$ for every $x \in B \setminus BN(\mathcal{E})$*

Then $R^1\pi_\mathcal{E}$ is free of rank k .*

Proof: By the previous theorem, $R^1\pi_*\mathcal{E}$ is torsion free of rank k . By [20] Prop 5.14, there is a covering of B by open sets U such that there exists an injective morphism $\alpha_U : R^1\pi_*\mathcal{E}|_U \rightarrow \mathcal{O}_U^k$. Besides, the global sections s define a sheaf morphism $\sigma : \mathcal{O}_B^k \rightarrow R^1\pi_*\mathcal{E}$ by

$$\begin{aligned} \sigma(V) : \quad \mathcal{O}_B^k(V) &\rightarrow R^1\pi_*\mathcal{E}(V) \\ (f_1, \dots, f_k) &\mapsto \sum_{i=1}^k f_i s_i \end{aligned}$$

for every open set $V \subset B$. Therefore $\varphi := \alpha \circ \sigma|_U : \mathcal{O}_U^k \rightarrow \mathcal{O}_U^k$ is an injective morphism outside an at least 2-codimensional analytic set. Therefore φ is an isomorphism, the exact sequence

$$0 \rightarrow \mathcal{O}_U^k \xrightarrow{\sigma} R^1\pi_*\mathcal{E}|_U \rightarrow \mathcal{Q} \rightarrow 0$$

has a retraction $r = \varphi^{-1} \circ \sigma : R^1\pi_*\mathcal{E}|_U \rightarrow \mathcal{O}_U^k$ therefore splits. Since \mathcal{Q} is a torsion sheaf we deduce from $R^1\pi_*\mathcal{E}|_U \simeq \mathcal{O}_U^k \oplus \mathcal{Q}$ that $\mathcal{Q} = 0$, hence $R^1\pi_*\mathcal{E}$ is locally free. \square

6.3 Infinitesimal logarithmic deformations: the hard part

We give in this section the proof of proposition 3.17.

Since $\sigma(0) = O_{n-1}$ is the intersection of two transversal rational curves which are contracted by F , there is a conjugation by a linear map φ (in particular birational) such that $\varphi^{-1}F\varphi = F' = \Pi'\sigma'$, satisfies

$$(S) \quad \frac{\partial \sigma'_1}{\partial z_2}(0) = \frac{\partial \sigma'_2}{\partial z_1}(0) = 0.$$

It means that $\sigma'^{-1}(C_{n-1})$ is tangent to $z_2 = 0$ and the other curve is tangent to $z_1 = 0$, therefore their strict transforms meet the exceptional curve C_0 respectively at $\{u' = v' = 0\}$ and $\{u = v = 0\}$.

Therefore in the following computations we shall suppose that the condition (S) is satisfied. Let $\Pi'' = \Pi_l \circ \Pi_{l+1} \circ \dots \circ \Pi_{n-1}$ be the composition of blowing-ups at the intersection of two curves and of Π_l , then it is the composition of mappings $(u, v) \mapsto (uv, v)$ or $(u', v') \mapsto (v', u'v')$, and of $\Pi_l(u', v') = (v' + a_{l-1}, u'v')$, hence

$$\Pi''(x, y) = (x^p y^q + a_{l-1}, x^r y^s)$$

where $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is the composition of matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, the last one being of the second type, therefore

$$\det \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \pm 1.$$

We have

$$\Pi''\sigma\Pi_0(u_0, v_0) = \Pi''(\sigma_1(u_0v_0, v_0), \sigma_2(u_0v_0, v_0)) = (\sigma_1^p\sigma_2^q(u_0v_0, v_0) + a_{l-1}, \sigma_1^r\sigma_2^s(u_0v_0, v_0)).$$

First case: there are at least two singular sequences, then

$$1 \leq p \leq r, \quad 1 \leq q \leq s, \quad p + q < r + s.$$

The jacobian is

$$D(\Pi''\sigma\Pi_0)(u_0, v_0) = \begin{pmatrix} v_0\sigma_1^{p-1}\sigma_2^{q-1}(u_0v_0, v_0)P(u_0, v_0) & \sigma_1^{p-1}\sigma_2^{q-1}(u_0v_0, v_0)Q(u_0, v_0) \\ v_0\sigma_1^{r-1}\sigma_2^{s-1}(u_0v_0, v_0)R(u_0, v_0) & \sigma_1^{r-1}\sigma_2^{s-1}(u_0v_0, v_0)S(u_0, v_0) \end{pmatrix}$$

where

$$\begin{cases} P(u, v) = p\sigma_2(uv, v)\partial_1\sigma_1(uv, v) + q\sigma_1(uv, v)\partial_1\sigma_2(uv, v), \\ Q(u, v) = p\sigma_2(uv, v)(u\partial_1\sigma_1(uv, v) + \partial_2\sigma_1(uv, v)) + q\sigma_1(uv, v)(u\partial_1\sigma_2(uv, v) + \partial_2\sigma_2(uv, v)) \\ R(u, v) = r\sigma_2(uv, v)\partial_1\sigma_1(uv, v) + s\sigma_1(uv, v)\partial_1\sigma_2(uv, v) \\ S(u, v) = r\sigma_2(uv, v)(u\partial_1\sigma_1(uv, v) + \partial_2\sigma_1(uv, v)) + s\sigma_1(uv, v)(u\partial_1\sigma_2(uv, v) + \partial_2\sigma_2(uv, v)) \end{cases}$$

For $i = 1, \dots, l-1$ we have also

$$D\Pi_i(u_i, v_i) = \begin{pmatrix} v_i & u_i \\ 0 & 1 \end{pmatrix}$$

In the local chart (u_i, v_i) containing O_i , for $i = 0, \dots, l-1$, X_i is tangent to $C_i = \{v_i = 0\}$, hence we have

$$X_i(u_i, v_i) = \begin{pmatrix} A_i(u_i, v_i) \\ v_i B_i(u_i, v_i) \end{pmatrix}.$$

For $i = 0, \dots, l-2$, we have at the point

$$(u_i, v_i) = \Pi_{i+1}(u_{i+1}, v_{i+1}) = (u_{i+1}v_{i+1} + a_i, v_{i+1}),$$

$$\begin{pmatrix} A_i(u_{i+1}v_{i+1} + a_i, v_{i+1}) \\ v_{i+1}B_i(u_{i+1}v_{i+1} + a_i, v_{i+1}) \end{pmatrix} - \begin{pmatrix} v_{i+1} & u_{i+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{i+1}(u_{i+1}, v_{i+1}) \\ v_{i+1}B_{i+1}(u_{i+1}, v_{i+1}) \end{pmatrix} = \begin{pmatrix} \alpha_i \\ 0 \end{pmatrix}$$

For $i = l-1$, at the point

$$(u_{l-1}, v_{l-1}) = \Pi'' \circ \sigma \circ \Pi_0(u_0, v_0) = \Pi''(\sigma_1(u_0v_0, v_0), \sigma_2(u_0v_0, v_0))$$

$$= (\sigma_1^p \sigma_2^q(u_0v_0, v_0) + a_{l-1}, \sigma_1^r \sigma_2^s(u_0v_0, v_0)),$$

$$\begin{pmatrix} A_{l-1}(\Pi'' \sigma \Pi_0(u_0, v_0)) \\ \sigma_1^r \sigma_2^s(u_0v_0, v_0) B_{l-1}(\Pi'' \sigma \Pi_0(u_0, v_0)) \end{pmatrix} - D(\Pi'' \sigma \Pi_0)(u_0, v_0) \begin{pmatrix} A_0(u_0, v_0) \\ v_0 B_0(u_0, v_0) \end{pmatrix} = \begin{pmatrix} \alpha_{l-1} \\ 0 \end{pmatrix}$$

Equivalently, we obtain

For $i = 0, \dots, l-2$,

$$(I_i) \quad A_i(u_{i+1}v_{i+1} + a_i, v_{i+1}) - v_{i+1} \{A_{i+1}(u_{i+1}, v_{i+1}) + u_{i+1}B_{i+1}(u_{i+1}, v_{i+1})\} = \alpha_i$$

$$(II_i) \quad B_i(u_{i+1}v_{i+1} + a_i, v_{i+1}) - B_{i+1}(u_{i+1}, v_{i+1}) = 0$$

For $i = l-1$, omitting subscripts,

$$(I_{l-1}) \quad A_{l-1}(\Pi'' \circ \sigma \circ \Pi_0(u, v)) - v \sigma_1^{p-1} \sigma_2^{q-1}(uv, v) \{P(u, v)A_0(u, v) + Q(u, v)B_0(u, v)\} = \alpha_{l-1}$$

$$(II_{l-1}) \quad \sigma_1 \sigma_2(uv, v) B_{l-1}(\Pi'' \circ \sigma \circ \Pi_0(u, v)) - v \{R(u, v)A_0(u, v) + S(u, v)B_0(u, v)\} = 0$$

For $i = 0, \dots, l-1$ and for $v_{i+1} = 0$ the equations (I_i) yield,

$$(1) \quad A_i(a_i, 0) = \alpha_i.$$

We put $u_i = t_i + a_i$, $i = 0, \dots, l-1$,

$$A_i(u_i, v_i) = A_i(a_i, 0) + A'_i(t_i, v_i) = A_i(a_i, 0) + \sum_{j+k>0} a_{j,k}^i t_i^j v_i^k,$$

$$B_i(u_i, v_i) = B_i(a_i, 0) + B'_i(t_i, v_i) = B_i(a_i, 0) + \sum_{j+k>0} b_{j,k}^i t_i^j v_i^k.$$

For $i = 0, \dots, l-2$, equations (II_i) give

$$(2) \quad B := B_0(a_0, 0) = \dots = B_{l-1}(a_{l-1}, 0),$$

$$(3) \quad B'_1(t_1, 0) = \cdots = B'_{l-1}(t_{l-1}, 0) = 0.$$

Replacing A_i and B_i by their expressions we have by (2),
For $i = 0, \dots, l-2$,

$$(I'_i) \quad A'_i((t_{i+1} + a_{i+1})v_{i+1}, v_{i+1}) - v_{i+1} \left\{ A_{i+1}(a_{i+1}, 0) + A'_{i+1}(t_{i+1}, v_{i+1}) \right. \\ \left. + (t_{i+1} + a_{i+1})[B + B'_{i+1}(t_{i+1}, v_{i+1})] \right\} = 0$$

$$(II'_i) \quad B'_i((t_{i+1} + a_{i+1})v_{i+1}, v_{i+1}) - B'_{i+1}(t_{i+1}, v_{i+1}) = 0$$

$$(I'_{l-1}) \quad \left\{ \begin{array}{l} A'_{l-1}(\sigma_1^p \sigma_2^q(uv, v), \sigma_1^r \sigma_2^s(uv, v)) \\ -v \sigma_1^{p-1} \sigma_2^{q-1}(uv, v) \left\{ P(u, v)[A_0(a_0, 0) + A'_0(t, v)] + Q(u, v)[B + B'_0(t, v)] \right\} \end{array} \right\} = 0$$

$$(II'_{l-1}) \quad \left\{ \begin{array}{l} \sigma_1 \sigma_2(uv, v)[B + B'_{l-1}(\sigma_1^p \sigma_2^q(uv, v), \sigma_1^r \sigma_2^s(uv, v))] \\ -v \left\{ R(u, v)[A_0(a_0, 0) + A'_0(t, v)] + S(u, v)[B + B'_0(t, v)] \right\} \end{array} \right\} = 0$$

Now, from the equations I'_i and II'_i , $0 \leq i \leq l-1$, we show that some terms vanish. In fact: For $i = 0, \dots, l-2$, we divide (I'_i) by v_{i+1} , we set $v_{i+1} = 0$, we apply (3), and we compare linear terms:

$$(4) \quad a_{10}^0 - a_{10}^1 = \cdots = a_{10}^{l-2} - a_{10}^{l-1} = B.$$

Equations (4) give

$$(5) \quad a_{10}^0 - a_{10}^{l-1} - (l-1)B = 0$$

For $i = 0, \dots, l-2$, we divide (II'_i) by v_{i+1} , we set $v_{i+1} = 0$, we apply (3):

$$(6) \quad b_{10}^0 a_1 + b_{01}^0 - b_{01}^1 = 0, \quad b_{01}^1 = \cdots = b_{01}^{l-1}.$$

$$(7) \quad b_{10}^1 = \cdots = b_{10}^{l-1} = 0.$$

Dividing (I'_{l-1}) by $v^2 \sigma_1^{p-1} \sigma_2^{q-1}(uv, v)$, setting $v = 0$, recalling that $\partial_1 \sigma_2(0) = \partial_2 \sigma_1(0) = 0$ and cancelling the factor $\partial_1 \sigma_1(0) \partial_2 \sigma_2(0) \neq 0$, we obtain

$$(8) \quad a_{1,0}^{l-1}(t + a_0) - \left\{ p[A_0(a_0, 0) + A'_0(t, 0)] + (p+q)(t + a_0)[B + B'_0(t, 0)] \right\} = 0$$

Constant part of (8) is

$$(9_c) \quad a_0 a_{10}^{l-1} - \{p A_0(a_0, 0) + (p+q) a_0 B\} = 0$$

Linear part of (8) is

$$(9_l) \quad p a_{10}^0 - a_{10}^{l-1} + (p+q)B + (p+q) a_0 b_{10}^0 = 0$$

Dividing (II'_{l-1}) by v^2 , setting $v = 0$ and cancelling the factor term $\partial_1 \sigma_1(0) \partial_2 \sigma_2(0) \neq 0$, we obtain

$$(10) \quad (t + a_0)B - r[A_0(a_0, 0) + A'_0(t, 0)] - (r+s)(t + a_0)[B + B'_0(t, 0)] = 0$$

Constant part of (10) is

$$(11_c) \quad rA_0(a_0, 0) + (r + s - 1)a_0B = 0$$

Linear part of (10) is

$$(11_l) \quad ra_{10}^0 + (r + s - 1)B + (r + s)a_0b_{10}^0 = 0$$

The determinant of the linear system (5), (9_c), (9_l), (11_c) and (11_l) with unknowns a_{10}^0 , a_{10}^{l-1} , $A_0(a_0, 0)$, B and b_{10}^0 is

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & -1 & 0 & -(l-1) & 0 \\ 0 & a_0 & -p & -(p+q)a_0 & 0 \\ p & -1 & 0 & (p+q) & (p+q)a_0 \\ 0 & 0 & r & (r+s-1)a_0 & 0 \\ r & 0 & 0 & (r+s-1) & (r+s)a_0 \end{vmatrix} \\ &= a_0^2(ps - qr) \left\{ (ps - qr) + 1 - (p + s) - rl \right\} \neq 0 \end{aligned}$$

Therefore, by (4), (6) and (7)

$$(12) \quad a_{10}^0 = \dots = a_{10}^{l-1} = B = 0, \quad b_{01}^0 = \dots = b_{01}^{l-1} \quad \text{and} \quad b_{10}^0 = \dots = b_{10}^{l-1} = 0.$$

Moreover we obtain

$$(13) \quad \boxed{\alpha_0 = A_0(a_0, 0) = 0.}$$

Second case: there is only one singular sequence s_m , $m \geq 1$, then

$$\begin{aligned} \begin{pmatrix} p & q \\ r & s \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{m-1} = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix} \\ \Pi'' \sigma \Pi_0(u_0, v_0) &= (\sigma_2(u_0 v_0, v_0) + a_{l-1}, \sigma_1 \sigma_2^m(u_0 v_0, v_0)) \\ D(\Pi'' \sigma \Pi_0)(u, v) &= \begin{pmatrix} vP(u, v) & Q(u, v) \\ v\sigma_2^{m-1}(uv, v)R(u, v) & \sigma_2^{m-1}(uv, v)S(u, v) \end{pmatrix} \end{aligned}$$

where

$$\begin{cases} P(u, v) &= \partial_1 \sigma_2(uv, v) \\ Q(u, v) &= u\partial_1 \sigma_2(uv, v) + \partial_2 \sigma_2(uv, v) \\ R(u, v) &= \sigma_2 \partial_1 \sigma_1(uv, v) + m\sigma_1 \partial_1 \sigma_2(uv, v) \\ S(u, v) &= \sigma_2(uv, v) \left(u\partial_1 \sigma_1(uv, v) + \partial_2 \sigma_1(uv, v) \right) \\ &\quad + m\sigma_1(uv, v) \left(u\partial_1 \sigma_2(uv, v) + \partial_2 \sigma_2(uv, v) \right) \end{cases}$$

The new equations are

$$\left\{ \begin{array}{l} (I'_{l-1}) \quad A'_{l-1}(\sigma_2(uv, v), \sigma_1 \sigma_2^m(uv, v)) \\ \quad -v \left\{ P(u, v)[A_0(a_0, 0) + A'_0(t, v)] + Q(u, v)[B + B'_0(t, v)] \right\} = 0 \\ \\ (II'_{l-1}) \quad \sigma_1 \sigma_2(uv, v) \left(B + B'_{l-1}(\sigma_2(uv, v), \sigma_1 \sigma_2^m(uv, v)) \right) \\ \quad -v \left\{ R(u, v)[A_0(a_0, 0) + A'_0(t, v)] + S(u, v)[B + B'_0(t, v)] \right\} = 0 \end{array} \right.$$

The end of the proof follows the same lines than in the first case. Details are left to the reader.

Remark 6. 73 *By induction it is possible to show that for any $k < r + s - (p + q)$, a similar Cramer system may be defined and that $\alpha_k = 0$. However, it is not possible to achieve the proof in this way because when $k = r + s - (p + q)$ a new unknown appears. This difficulty is explained by the fact that in general **there is a relation** among the θ^i 's, as we shall see in the sequel.*

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